



# Weak Forms Of Open Sets In An Ideal Binary Topological Space

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**Abstract.** In this work, we introduce and study the new concepts called  $\alpha$ -*bl*-open sets, semi-*bl*-open sets, pre-*bl*-open sets,  $b$ - $I_b$ -open sets and  $\beta$ -*bl*-open sets, which are weak forms of binary open sets in an ideal binary topological spaces. Also we characterize the relations between them and the related properties. And the independence of *b*-open sets and *bl*-open sets is established.

**Key words:**  $\alpha$  s. -*bl*-open sets, semi-*bl*-open sets, pre-*bl*-open sets,  $b$ - $I_b$ -open sets and  $\beta$ -*bl*-open sets.

## 1 Introduction and Preliminaries

In classical topology, Njastad[16] studied the characteristics of  $\alpha$ -open sets and showed that they constitute a topology. The notion of a semi-open set and its topological characteristics was first introduced by Levine[12]. Mashhour, et al.[14] examined the characteristics of pre-continuous functions and presented the idea of pre-open sets. Additionally,  $\alpha$ -continuous and  $\alpha$ -open mappings were examined by Mashhour, et al.[15]. Abd Monsef, et al. $\beta$ -open sets and  $\beta$ -continuous functions were introduced and described in[2]. The relationship between  $\alpha$ -open sets, semi-open sets, and pre-open sets was covered by Reilly et al.[18]. Andrijević [1] introduced and investigated a new generalized open sets in a topological space, called *b*-open sets. Kuratowski [10, 11] treats ideals in a topological space  $(X, \tau)$ . An ideal space is a topological space that has an ideal  $\mathcal{I}$  and is represented by  $(X, \tau, \mathcal{I})$ . Additionally, he defined the local function with regard to an ideal  $\mathcal{I}$  and  $\tau$  for every subset of  $X$ .  $A^*(\mathcal{I}, \tau)$  or just  $A^*$  represents the local function of a subset  $A$  of  $X$  with respect to  $\mathcal{I}$  and  $\tau$ .

Additionally, Vaidyanathaswamy [21, 22] expanded the study of ideals and local functions in 1945. Vaidyanathaswamy [22], Hashimoto [7, 8], Hayashi [6], and Samuels [19] have examined the properties of the topology formed by the ideal  $\mathcal{I}$  and  $\tau$ , known as the  $\star$ -topology which is finer than  $\tau$ , indicated by  $\tau^*$ . In addition to their research, Jankovic and Hamlett [9] compiled all of the findings from the aforementioned studies in 1990. Completely codense ideals and  $\mathcal{I}$ -dense sets were introduced and investigated by Dontchev, Ganster, and Rose [5] in 1999. *I*-open sets and *I*-continuous functions were introduced by Abd El-monsef, Lashien, and Nasef [3]. Shyamapada Modak et al.[20] Following that, in addition to presenting some observations on ideal binary topological spaces, we investigate the relationships among binary topological spaces. In this study, we define a few fundamental characteristics and attributes, emphasizing interior and closure, and we determine the connections between them.

**Definition 1.1** [20] Let  $X$  and  $Y$  be any two non empty sets. A binary ideal from  $X$  to  $Y$  is a binary structure  $\mathcal{I} \subseteq \wp(X) \times \wp(Y)$  that satisfies the following axioms:

1.  $(A, B) \in \mathcal{I}$  and  $(C, D) \subseteq (A, B)$  implies  $(C, D) \in \mathcal{I}$  (hereditary).
2.  $(A_1, B_1) \in \mathcal{I}$  and  $(A_2, B_2) \in \mathcal{I}$  implies  $(A_1 \cup A_2, B_1 \cup B_2) \in \mathcal{I}$  finite additivity).

**Definition 1.2** [20] An binary ideal  $\mathcal{I}$  on  $\wp(X) \times \wp(Y)$  is called ideal binary topological space and it is denoted as  $(X, Y, \mathcal{M}, \mathcal{I})$ . For a binary subset  $(A, B)$  of  $X \times Y$ , we define the following set operator:  $(,)^*: \wp(X) \times \wp(Y) \rightarrow \wp(X) \times \wp(Y)$ , is called a binary local function with respect to  $\mathcal{M}$  and  $\mathcal{I}$  is defined as follows:  $(A, B)^*(\mathcal{I}, \mathcal{M}) = \{(x, y) \in (X, Y) : (U \cap A, V \cap B) \notin \mathcal{I} \text{ for every } (U, V) \in \mathcal{M}(x, y) \text{ where } \mathcal{M}(x, y) = \{(U, V) \in \mathcal{M} : (x, y) \in (U, V)\}\}$ . In case there is no confusion  $(A, B)^*(\mathcal{I}, \mathcal{M})$  is briefly denoted by  $(A, B)^*$  and is called Binary local

function of  $(A, B)$  with respect to  $\mathcal{I}$  and  $\mathcal{M}$ . From [20], we have  $C^*: \wp(X) \times \wp(Y) \rightarrow \wp(X) \times \wp(Y)$  is a Kuratowski closure operator. Therefore  $\{(U, V) \subseteq (X, Y): C^*[(X, Y) \setminus (U, V)] = (X, Y) \setminus (U, V)\}$  forms a binary topology on  $X \times Y$ , and it is denoted as  $\mathcal{M}^*$ .

**Theorem 1.3** [20] Let  $(X, Y, \mathcal{M}, \mathcal{I})$  be an ideal binary topological space. Then  $\beta(\mathcal{M}, \mathcal{I}) = (V_1, V_2) \setminus I: (V_1, V_2)$  is a binary open set of  $(X, Y, \mathcal{M}), I \in \mathcal{I}$  is a basis for  $\mathcal{M}^*$ .

**Theorem 1.4** [4] Let  $(\mathcal{B}^{\mathcal{M}}, I)$  be a space and  $(A, B)$  and  $(C, D)$  subsets of  $\mathcal{M}$ . Then

1.  $(A, B) \subseteq (C, D) \Rightarrow (A, B)_b^* \subseteq (C, D)_b^*$ ,
2.  $I \subseteq I' \Rightarrow (A, B)_b^*(I') \subseteq (A, B)_b^*(I)$ ,
3.  $(A, B)_b^* = b-cl(A, B)_b^* \subseteq b-cl(A, B)[(A, B)_b^* \text{ is } b\text{-closed of } b-cl(A, B)]$ ,
4.  $((A, B)_b^*)_b^* \subseteq (A, B)_b^*$ ,
5.  $(A, B)_b^* \cup (C, D)_b^* = ((A, B) \cup (C, D))_b^*$ ,
6.  $(U, V) \in \mathcal{M} \Rightarrow (U, V) \cap (A, B)_b^* = (U, V) \cap ((U, V) \cap (A, B))_n^* \subseteq ((U, V) \cap (A, B))_b^*$ .

**Definition 1.5** [4] Let  $\mathcal{B}^{\mathcal{M}}$  be a binary topological space with an ideal  $I$  on  $(X, Y)$ . The operator  $b-cl^*$  is called a binary  $\star$ -closure and is defined as  $b-cl^*(A, B) = (A, B) \cup (A, B)_b^*$  for  $A \subseteq (X, Y)$ .

**Theorem 1.6** [4] The set operator  $b-cl^*$  satisfies the following Properties:

1.  $(A, B) \subseteq b-cl^*(A, B)$ ,
2.  $b-cl^*(\phi, \phi) = \phi$  and  $b-cl^*(X, Y) = (X, Y)$ ,
3. If  $(A, B) \subset (C, D)$  then  $b-cl^*(A, B) \subseteq b-cl^*(C, D)$ ,
4.  $b-cl^*(A, B) \cup b-cl^*(C, D) = b-cl^*((A, B) \cup (C, D))$ ,
5.  $b-cl^*(b-cl^*(A, B)) = b-cl^*(A, B)$ .

Now,  $\mathcal{M}^*(I, \mathcal{M}) = \{(U, V) \subseteq (X, Y): b-cl^*((X, Y) - (U, V)) = (X, Y) - (U, V)\}$ .  $\mathcal{M}^*(I, \mathcal{M})$  is called binary  $\star$ -topology which is finer than  $\mathcal{M}$  (we can write  $\mathcal{M}^*$  for  $\mathcal{M}^*(I, \mathcal{M})$ ). The elements of  $\mathcal{M}^*(I, \mathcal{M})$  are called binary  $\star$ -open (simply,  $b \star$ -open) and the complement of an  $b \star$ -open is called binary  $b \star$ -closed (simply  $b \star$ -closed). Here  $b-cl^*(A, B)$  and  $b-int^*(A, B)$  will denote the closure and interior of  $(A, B)$  in  $((X, Y), \mathcal{M}^*)$ . Nithyanantha Jothi, et al.[17] introduced topology between two sets and also studied some of their properties. Topology between two sets is the binary structure from  $X$  to  $Y$  which is defined to be the ordered pairs  $(A, B)$  where  $A \subseteq X$  and  $B \subseteq Y$ .

Let  $X$  and  $Y$  be any two nonempty sets. A binary topology [17] from  $X$  to  $Y$  is a binary structure  $\mathcal{M} \subseteq \mathbb{P}(X) \times \mathbb{P}(Y)$  that satisfies the axioms namely

1.  $(\phi, \phi)$  and  $(X, Y) \in \mathcal{M}$ ,
2.  $(A_1 \cap A_2, B_1 \cap B_2) \in \mathcal{M}$  whenever  $(A_1, B_1) \in \mathcal{M}$  and  $(A_2, B_2) \in \mathcal{M}$ , and
3. If  $\{(A_\alpha, B_\alpha): \alpha \in \delta\}$  is a family of members of  $\mathcal{M}$ , then  $(\bigcup_{\alpha \in \delta} A_\alpha, \bigcup_{\alpha \in \delta} B_\alpha) \in \mathcal{M}$ .

If  $\mathcal{M}$  is a binary topology from  $X$  to  $Y$  then the triplet  $(X, Y, \mathcal{M})$  is called a binary topological space and the members of  $\mathcal{M}$  are called the binary open subsets of the binary topological space  $(X, Y, \mathcal{M})$ . The elements of  $X \times Y$  are called the binary points of the binary topological space  $(X, Y, \mathcal{M})$ . If  $Y = X$  then  $\mathcal{M}$  is called a binary topology on  $X$  in which case we write  $(X, \mathcal{M})$  as a binary topological space.

**Definition 1.7** [17] Let  $X$  and  $Y$  be any two nonempty sets and let  $(A, B)$  and  $(C, D) \in \mathbb{P}(X) \times \mathbb{P}(Y)$ . We say that  $(A, B) \subseteq (C, D)$  if  $A \subseteq C$  and  $B \subseteq D$ .

**Definition 1.8** [17] Let  $(X, Y, \mathcal{M})$  be a binary topological space and  $A \subseteq X, B \subseteq Y$ . Then  $(A, B)$  is called binary closed in  $(X, Y, \mathcal{M})$  if  $(X \setminus A, Y \setminus B) \in \mathcal{M}$ .

**Proposition 1.9** [17] Let  $(X, Y, \mathcal{M})$  be a binary topological space and  $(A, B) \subseteq (X, Y)$ .

Let  $(A, B)^{1*} = \bigcap \{A_\alpha: (A_\alpha, B_\alpha) \text{ is binary closed and } (A, B) \subseteq (A_\alpha, B_\alpha)\}$  and  $(A, B)^{2*} = \bigcap \{B_\alpha: (A_\alpha, B_\alpha) \text{ is binary closed and } (A, B) \subseteq (A_\alpha, B_\alpha)\}$ . Then  $((A, B)^{1*}, (A, B)^{2*})$  is binary closed and  $(A, B) \subseteq ((A, B)^{1*}, (A, B)^{2*})$ .

**Proposition 1.10** [17] Let  $(X, Y, \mathcal{M})$  be a binary topological space and  $(A, B) \subseteq (X, Y)$ . Let  $(A, B)^{1*} = \bigcup \{A_\alpha: (A_\alpha, B_\alpha) \text{ is binary open and } (A_\alpha, B_\alpha) \subseteq (A, B)\}$  and  $(A, B)^{2*} = \bigcup \{B_\alpha: (A_\alpha, B_\alpha) \text{ is binary open and } (A_\alpha, B_\alpha) \subseteq (A, B)\}$ .

**Definition 1.11** [17] The ordered pair  $((A, B)^{1*}, (A, B)^{2*})$  is called the binary closure of  $(A, B)$ , denoted by  $b-cl(A, B)$  in the binary space  $(X, Y, \mathcal{M})$  where  $(A, B) \subseteq (X, Y)$ .

**Definition 1.12** [17] The ordered pair  $((A, B)^{1*}, (A, B)^{2*})$  defined in proposition 1.10 is called the binary interior of  $(A, B)$ , denoted by  $b-int(A, B)$ . Here  $((A, B)^{1*}, (A, B)^{2*})$  is binary open and  $((A, B)^{1*}, (A, B)^{2*}) \subseteq$

$(A, B)$ .

**Definition 1.13** [17] Let  $(X, Y, \mathcal{M})$  be a binary topological space and let  $(x, y) \subseteq (X, Y)$ . The binary open set  $(A, B)$  is said to be a binary neighbourhood of  $(x, y)$  if  $x \in A$  and  $y \in B$ .

**Proposition 1.14** [17] Let  $(A, B) \subseteq (C, D) \subseteq (X, Y)$  and  $(X, Y, \mathcal{M})$  be a binary topological space. Then, the following statements hold:

1.  $b\text{-int}(A, B) \subseteq (A, B)$ .
2. If  $(A, B)$  is binary open, then  $b\text{-int}(A, B) = (A, B)$ .
3.  $b\text{-int}(A, B) \subseteq b\text{-int}(C, D)$ .
4.  $b\text{-int}(b\text{-int}(A, B)) = b\text{-int}(A, B)$ .
5.  $(A, B) \subseteq b\text{-cl}(A, B)$ .
6. If  $(A, B)$  is binary closed, then  $b\text{-cl}(A, B) = (A, B)$ .
7.  $b\text{-cl}(A, B) \subseteq b\text{-cl}(C, D)$ .
8.  $b\text{-cl}(b\text{-cl}(A, B)) = b\text{-cl}(A, B)$ .

In future an ideal binary topological spaces  $(X, Y, \mathcal{M}, I)$  is referred as a space  $(\mathcal{B}^{\mathcal{M}}, I)$ .

## 2 Weak forms of open sets in $(\mathcal{B}^{\mathcal{M}}, I)$

**Definition 2.1** A subset  $(A, B)$  of space  $(\mathcal{B}^{\mathcal{M}}, I)$  is said to be

1. binary  $\alpha$ - $I$ -open (briefly  $\alpha$ - $bl$ -open) if  $(A, B) \subseteq b\text{-int}(b\text{-cl}^*(b\text{-int}(A, B)))$ ,
2. binary semi- $I$ -open (briefly semi- $bl$ -open) if  $(A, B) \subseteq b\text{-cl}^*(b\text{-int}(A, B))$ ,
3. binary pre- $I$ -open (briefly pre- $bl$ -open) if  $(A, B) \subseteq b\text{-int}(b\text{-cl}^*(A, B))$ ,
4. binary  $b$ - $I$ -open (briefly  $b$ - $I_b$ -open) if  $(A, B) \subseteq b\text{-int}(b\text{-cl}^*(A, B)) \cup b\text{-cl}^*(b\text{-int}(A, B))$ ,
5. binary  $\beta$ - $I$ -open (briefly  $\beta$ - $bl$ -open) if  $(A, B) \subseteq b\text{-cl}^*(b\text{-int}(b\text{-cl}^*(A, B)))$ .

The complements of the above mentioned sets are called their respective closed sets.

**Theorem 2.2** For a subset  $(A, B)$  in a space  $(\mathcal{B}^{\mathcal{M}}, I)$  the following relations hold;

1.  $(A, B)$  is  $bl$ -open  $\Rightarrow (A, B)$  is  $\alpha$ - $bl$ -open,
2.  $(A, B)$  is  $\alpha$ - $bl$ -open  $\Rightarrow (A, B)$  is semi- $bl$ -open,
3.  $(A, B)$  is  $\alpha$ - $bl$ -open  $\Rightarrow (A, B)$  is pre- $bl$ -open,
4.  $(A, B)$  is semi- $bl$ -open  $\Rightarrow (A, B)$  is  $b$ - $I_b$ -open,
5.  $(A, B)$  is pre- $bl$ -open  $\Rightarrow (A, B)$  is  $b$ - $I_b$ -open,
6.  $(A, B)$  is  $b$ - $I_b$ -open  $\Rightarrow (A, B)$  is  $\beta$ - $bl$ -open.

Proof. 1.  $(A, B)$  is  $b$ -open  $\Rightarrow (A, B) = b\text{-int}(A, B)$ . But  $(A, B) \subseteq b\text{-cl}^*(A, B) = b\text{-cl}^*(b\text{-int}(A, B)) \subseteq b\text{-cl}^*(b\text{-int}(b\text{-cl}^*(A, B)))$  this proves that  $(A, B)$  is  $\alpha$ - $bl$ -open.

2.  $(A, B)$  is  $\alpha$ - $bl$ -open  $\Rightarrow (A, B) \subseteq b\text{-int}(b\text{-cl}^*(b\text{-int}(A, B))) \subseteq b\text{-cl}^*(b\text{-int}(A, B))$  which proves that  $(A, B)$  is semi- $bl$ -open.

3.  $(A, B)$  is  $\alpha$ - $bl$ -open  $\Rightarrow (A, B) \subseteq b\text{-int}(b\text{-cl}^*(b\text{-int}(A, B))) \subseteq b\text{-int}(b\text{-cl}^*(A, B))$  which proves that  $(A, B)$  is pre- $bl$ -open.

4.  $(A, B)$  is semi- $bl$ -open  $\Rightarrow (A, B) \subseteq b\text{-cl}^*(b\text{-int}(A, B)) \subseteq b\text{-cl}^*(b\text{-int}(A, B)) \cup b\text{-int}(b\text{-cl}^*(A, B))$  which proves that  $(A, B)$  is  $b$ - $I_b$ -open.

5.  $(A, B)$  is pre- $bl$ -open  $\Rightarrow (A, B) \subseteq b\text{-int}(b\text{-cl}^*(A, B)) \subseteq b\text{-int}(b\text{-cl}^*(A, B)) \cup b\text{-cl}^*(b\text{-int}(A, B))$  which proves that  $(A, B)$  is  $b$ - $I_b$ -open.

6.  $(A, B)$  is  $b$ - $I_b$ -open  $\Rightarrow (A, B) \subseteq b\text{-int}(b\text{-cl}^*(A, B)) \cup b\text{-cl}^*(b\text{-int}(A, B)) \subseteq b\text{-cl}^*(b\text{-int}(b\text{-cl}^*(A, B))) \cup b\text{-cl}^*(b\text{-int}(b\text{-cl}^*(A, B))) = b\text{-cl}^*(b\text{-int}(b\text{-cl}^*(A, B)))$  which proves that  $(A, B)$  is  $\beta$ - $bl$ -open.

**Remark 2.3** These relations are shown in the diagram.

|                            |               |                                                 |               |                                                |
|----------------------------|---------------|-------------------------------------------------|---------------|------------------------------------------------|
|                            |               |                                                 |               |                                                |
| <b><math>b</math>-open</b> | $\rightarrow$ | <b><math>\alpha</math>-<math>bl</math>-open</b> | $\rightarrow$ | <b>pre-<math>bl</math>-open</b>                |
|                            |               | $\downarrow$                                    |               | $\downarrow$                                   |
|                            |               | <b>semi-<math>bl</math>-open</b>                | $\rightarrow$ | <b><math>b</math>-<math>I_b</math>-open</b>    |
|                            |               |                                                 |               | $\downarrow$                                   |
|                            |               |                                                 |               | <b><math>\beta</math>-<math>bl</math>-open</b> |

The converses of each statement in Theorem 2.2 are need not to be true as shown in the following Example.

**Example 2.4** Let  $X = \{a, b\}$ ,  $Y = \{1, 2\}$  and  $\mathcal{M} = \{(\phi, \phi), (\phi, \{1\}), (\{a, \}, \{2\}), (\{a\}, Y), (X, Y)\}$ . Let the ideal

be  $\mathcal{I} = \{(\phi, \phi), (\phi, \{1\})\}$ . Then the subset

1.  $(X, \{2\})$  is  $\alpha$ - $bl$ -open but not  $b$ -open.
2.  $(\{a\}, \{1\})$  is pre- $bl$ -open but not  $\alpha$ - $bl$ -open.
3.  $(\phi, \{2\})$  is  $b$ - $I_b$ -open but not semi- $bl$ -open.
4.  $(\phi, \{1\})$  is  $b$ - $I_b$ -open but not pre- $bl$ -open.

**Example 2.5** Let  $X = \{a, b\}$ ,  $Y = \{1, 2\}$

and  $\mathcal{M} = \{(\phi, \phi), (\phi, \{2\}), (\{a\}, \{1\}), (\{a\}, Y), (X, \{1\}), (X, Y)\}$ .

Let the ideal be  $\mathcal{I} = \{(\phi, \phi), (\{a\}, \phi), (\{b\}, \phi), (X, \phi)\}$ . Then the subset

1.  $(\{a\}, \{2\})$  is semi- $bl$ -open but not  $\alpha$ - $bl$ -open.
2.  $(\{b\}, \{1\})$  is  $\beta$ - $bl$ -open but not  $b$ - $I_b$ -open.

**Remark 2.6** In Remark 2.3 relationship between these various open sets are given.

Here an example of a space where all these converse implication also hold good is given, In Example 2.5. In the space  $(\mathcal{B}^{\mathcal{M}}, I)$ ,  $(\{a\}, \phi)$  is not  $\beta$ - $bl$ -open for  $b-cl^*(\{a\}, \phi) = (\{a\}, \phi)$  and  $b-int(b-cl^*(\{a\}, \phi)) = (\phi, \phi)$ . Then  $(\{b\}, \{1\})$  is not  $b$ - $I_b$ -open, not pre- $bl$ -open, not semi- $bl$ -open, not  $\alpha$ - $bl$ -open and  $b$ -open. But  $(A, B) = (\{b\}, \{1\})$  is  $\beta$ - $bl$ -open and hence  $(A, B) = (\{b\}, \{1\})$  is only  $\beta$ - $bl$ -open subset. Then it's  $b$ - $I_b$ -open, pre- $bl$ -open, semi- $bl$ -open,  $\alpha$ - $bl$ -open and  $b$ -open. thus all the converse implications also hold good in this space.

**Remark 2.7** The family of  $b$ -open sets and the family of  $bl$ -open sets are independent in  $(\mathcal{B}^{\mathcal{M}}, I)$ .

**Example 2.8** Let  $X = \{a, b, c\}$ ,  $Y = \{1, 2\}$

and  $\mathcal{M} = \{(\phi, \phi), (\{a\}, \{1\}), (\{b\}, \phi), (\{b\}, \{2\}), (\{a, b\}, \{1\}), (\{a, b\}, Y), (X, Y)\}$ .

1.  $(\{a, c\}, \{1\})$  is  $bl$ -open but not  $b$ -open.
2.  $(\{b\}, \{2\})$  is  $b$ -open but not  $bl$ -open.

**Theorem 2.9** A subset  $(A, B)$  of a space  $(\mathcal{B}^{\mathcal{M}}, I)$  is  $\alpha$ - $bl$ -open  $\Leftrightarrow (A, B)$  is semi- $bl$ -open and pre- $bl$ -open.

Proof.  $\Rightarrow$  Part follows from (2) and (3) of Theorem 2.2.

$\Leftarrow$  If  $(A, B)$  is semi- $bl$ -open and pre- $bl$ -open then  $(A, B) \subseteq b-int(b-cl^*(A, B))$  and  $(A, B) \subseteq b-cl^*(b-int(A, B))$ .

Thus  $(A, B) \subseteq b-int(b-cl^*(A, B)) \subseteq b-int(b-cl^*(b-cl^*(b-int(A, B)))) = b-int(b-cl^*(b-int(A, B)))$  which proves that  $(A, B)$  is  $\alpha$ - $bl$ -open.

**Remark 2.10** In a space  $(\mathcal{B}^{\mathcal{M}}, I)$ , the family of semi- $bl$ -open sets and the family of pre- $bl$ -open sets are independent of each other as shown in the following Example.

Except this there is no other independent.

**Remark 2.11** From Theorem 2.9 and Remark 2.10 we get a decomposition of  $\alpha$ - $bl$ -open into a semi- $bl$ -open and pre- $bl$ -open set

**Example 2.12** In Example 2.4, then the subset

1.  $(\phi, \{1\})$  is semi- $bl$ -open but not pre- $bl$ -open.
2.  $(\{b\}, \{2\})$  is pre- $bl$ -open but not semi- $bl$ -open.

**Theorem 2.13** If a subset  $(A, B)$  of a space  $(\mathcal{B}^{\mathcal{M}}, I)$  is both  $b$   $\star$ -closed and  $\beta$ - $bl$ -open, then  $(A, B)$  is semi- $bl$ -open.

Proof. Since  $(A, B)$  is  $\beta$ - $bl$ -open,  $(A, B) \subseteq b-cl^*(b-int(b-cl^*(A, B))) = b-cl^*(b-int(A, B))$ ,  $(A, B)$  being  $b$   $\star$ -closed. Therefore  $(A, B)$  is semi- $bl$ -open.

**Theorem 2.14** A subset  $(A, B)$  of a space  $(\mathcal{B}^{\mathcal{M}}, I)$  is semi- $bl$ -open if and only if  $b-cl^*(A, B) = b-cl^*(b-int(A, B))$ .

Proof. Let  $(A, B)$  be semi- $bl$ -open. Then  $(A, B) \subseteq b-cl^*(b-int(A, B))$  and  $b-cl^*(A, B) \subseteq b-cl^*(b-int(A, B))$ . But  $b-cl^*(b-int(A, B)) \subseteq b-cl^*(A, B)$ . Thus  $b-cl^*(A, B) = b-cl^*(b-int(A, B))$ .

Conversely, let the condition hold. We have  $(A, B) \subseteq b-cl^*(A, B) = b-cl^*(b-int(A, B))$ , by assumption. Thus  $(A, B) \subseteq b-cl^*(b-int(A, B))$  and hence  $(A, B)$  is semi- $bl$ -open.

**Proposition 2.15** In  $(\mathcal{B}^{\mathcal{M}}, I)$  if  $(A, B)$  is a  $b$ - $I_b$ -open set such that  $b-cl^*(A, B) = (\phi, \phi) = \phi$ , then  $(A, B)$  is semi- $bl$ -open.

**Theorem 2.16** A subset  $(A, B)$  of a space  $(\mathcal{B}^{\mathcal{M}}, I)$  is semi- $bl$ -open if and only if there exists a  $b$ -open set  $(G, H)$  such that  $(G, H) \subseteq (A, B) \subseteq b-cl^*(G, H)$ .

Proof. Let  $(A, B)$  be semi- $bl$ -open. Then  $(A, B) \subseteq b-cl^*(b-int(A, B))$ . Take  $b-int(A, B) = (G, H)$ . Then

$(G, H) \subseteq (A, B) \subseteq b-cl^*(G, H)$ , where  $(G, H)$  is  $b$ -open.

Conversely, let  $(G, H) \subseteq (A, B) \subseteq b-cl^*(G, H)$  for some  $b$ -open set  $(G, H)$ . Since  $(G, H) \subseteq (A, B)$ ,  $(G, H) \subseteq b-int(A, B)$  and  $(A, B) \subseteq b-cl^*(G, H) \subseteq b-cl^*(b-int(A, B))$  which implies  $(A, B)$  is semi- $bI$ -open.

**Theorem 2.17** If  $(A, B)$  is a semi- $bI$ -open set in a space  $(\mathcal{B}^M, I)$  and  $(A, B) \subseteq (B, C) \subseteq b-cl^*(A, B)$ , then  $(B, C)$  is semi- $bI$ -open.

Proof. By assumption  $(B, C) \subseteq b-cl^*(A, B) \subseteq b-cl^*(b-cl^*(b-int(A, B)))$  (for  $(A, B)$  is semi- $bI$ -open)  $= b-cl^*(b-int(A, B)) \subseteq b-cl^*(b-int(B, C))$  by assumption. This implies  $(B, C)$  is semi- $bI$ -open.

**Theorem 2.18** In a space  $(\mathcal{B}^M, I)$ , for a subset  $(A, B)$ , the following results hold;

1.  $(A, B)$  is  $bI$ -open  $\Rightarrow (A, B)$  is pre- $bI$ -open,
2.  $(A, B)$  is  $bI$ -open  $\Rightarrow (A, B)$  is  $\beta$ - $bI$ -open,
3.  $(A, B)$  is  $bI$ -open  $\Rightarrow (A, B)$  is  $b-I_b$ -open.

Proof. 1.  $(A, B)$  is  $bI$ -open  $\Rightarrow (A, B) \subseteq b-int((A, B)_b^*) \subseteq b-int(b-cl^*(A, B))$  which proves that  $(A, B)$  is pre- $bI$ -open.

2.  $(A, B)$  is  $bI$ -open  $\Rightarrow (A, B) \subseteq b-int((A, B)_b^*) \subseteq b-cl^*(b-int(b-cl^*(A, B)))$  which proves that  $(A, B)$  is  $\beta$ - $bI$ -open.

3.  $(A, B)$  is  $bI$ -open  $\Rightarrow (A, B) \subseteq b-int(A, B)_b^* \subseteq b-int(b-cl^*(A, B)) \subseteq b-int(b-cl^*(A, B)) \cup n-cl^*(n-int(A, B))$  which proves that  $(A, B)$  is  $b-nI$ -open.

**Remark 2.19** The converses of (1), (2) and (3) in Theorem 2.18 are need not to be true as shown in the following Example.

**Example 2.20** In Example 2.8, then the subset

1.  $(\{b\}, \{2\})$  is pre- $bI$ -open but not  $bI$ -open and  $b-I_b$ -open but not  $bI$ -open.
2.  $(\{b\}, \{2\})$  is  $\beta$ - $bI$ -open but not  $bI$ -open.

**Remark 2.21**

1. In a space  $(\mathcal{B}^M, I)$ , the family of  $bI$ -open sets and the family of  $\alpha$ - $bI$ -open sets are independent of each other.

2. In a space  $(\mathcal{B}^M, I)$ , the family of  $bI$ -open sets and the family of semi- $bI$ -open sets are independent of each other.

**Example 2.22** In Example 2.4, then the subset

1.  $(\{a\}, \{1\})$  is  $bI$ -open but not  $\alpha$ - $bI$ -open.
2.  $(\phi, \{1\})$  is  $\alpha$ - $bI$ -open but not  $bI$ -open.

Examples (1) and (2) verify (1) of Remark 2.21.

3.  $(\{b\}, Y)$  is  $bI$ -open but not semi- $bI$ -open.

4.  $(\phi, \{1\})$  is semi- $bI$ -open but not  $bI$ -open.

Examples (3) and (4) verify (2) of Remark 2.21.

**Proposition 2.23** For a subset of  $(A, B)$  a space  $(\mathcal{B}^M, I)$ , the following properties hold;

1.  $(A, B)$  is  $\alpha$ - $bI$ -open  $\Rightarrow (A, B)$  is  $b\alpha$ -open,
2.  $(A, B)$  is pre- $bI$ -open  $\Rightarrow (A, B)$  is  $bp$ -open,
3.  $(A, B)$  is  $b-I_b$ -open  $\Rightarrow (A, B)$  is binary  $b$ -open,
4.  $(A, B)$  is  $\beta$ - $bI$ -open  $\Rightarrow (A, B)$  is binary  $b$ -open.

Proof. 1. Let  $(A, B)$  be a  $\alpha$ - $bI$ -open set. Then  $(A, B) \subseteq b-int(b-cl^*(b-int(A, B))) \subseteq b-int(b-cl(b-int(A, B)))$ . This shows that  $(A, B)$  is  $b\alpha$ -open.

2. Let  $(A, B)$  be a pre- $bl$ -open set. Then  $(A, B) \subseteq b\text{-int}(b\text{-cl}^*(A, B)) \subseteq b\text{-int}(b\text{-cl}(A, B))$ . This shows that  $(A, B)$  is  $bp$ -open.

3. Let  $(A, B)$  be a  $b\text{-}I_b$ -open set. Then  $(A, B) \subseteq b\text{-int}(b\text{-cl}^*(A, B)) \cup b\text{-cl}^*(b\text{-int}(A, B)) \subseteq b\text{-int}(b\text{-cl}(A, B)) \cup b\text{-cl}(b\text{-int}(A, B))$ . This shows that  $(A, B)$  is binary  $b$ -open.

4. Let  $(A, B)$  be a  $\beta$ - $bl$ -open set. Then  $(A, B) \subseteq b\text{-cl}^*(b\text{-int}(b\text{-cl}^*(A, B))) \subseteq b\text{-cl}(b\text{-int}(b\text{-cl}(A, B)))$ . This shows that  $(A, B)$  is  $b\beta$ -open.

**Remark 2.24** The converses of Proposition 2.23 are need not to be true in general as shown in the following Example.

**Example 2.25** Let  $X = \{a, b\}$ ,  $Y = \{1, 2\}$  and  $\mathcal{M} = \{(\phi, \phi), (\phi, \{1\}), (\{a\}, \{1\}), (X, Y)\}$ . Let the ideal be  $\mathcal{I} = \{\mathcal{P}(X) \times \mathcal{P}(Y)\}$ . Then the subset

1.  $(\{a\}, Y)$  is  $b\alpha$ -open but not  $\alpha$ - $bl$ -open.
2.  $(\{b\}, \{1\})$  is  $bp$ -open but not pre- $bl$ -open.
3.  $(\phi, Y)$  is binary  $b$ -open but not  $b\text{-}I_b$ -open.
4.  $(X, \{1\})$  is  $b\beta$ -open but not  $\beta$ - $bl$ -open.

**Lemma 2.26** Let  $(\mathcal{B}^{\mathcal{M}}, \mathcal{I})$  be a space and  $(A, B)$  a subset of  $\mathcal{B}^{\mathcal{M}}$ . If  $(G, H)$  is  $b$ -open in  $(\mathcal{B}^{\mathcal{M}}, \mathcal{I})$ , then  $(G, H) \cap b\text{-cl}^*(A, B) \subseteq b\text{-cl}^*((G, H) \cap (A, B))$ .

Proof.  $(G, H) \cap b\text{-cl}^*(A, B) = (G, H) \cap (A, B)_b^* \cup (A, B) = (G, H) \cap (A, B)_b^* \cup (G, H) \cap (A, B) \subseteq (G, H) \cap (A, B)_b^* \cup (G, H) \cap (A, B)$  by (5) of Theorem 1.4. Thus  $(G, H) \cap b\text{-cl}^*(A, B) \subseteq (G, H) \cap (A, B)_b^* \cup (G, H) \cap (A, B) = b\text{-cl}^*((G, H) \cap (A, B))$ .

**Proposition 2.27** The intersection of a pre- $bl$ -open set and a  $b$ -open set is pre- $bl$ -open.

Proof. Let  $(A, B)$  be pre- $bl$ -open and  $(G, H)$  be  $b$ -open. Then  $(A, B) \subseteq b\text{-int}(b\text{-cl}^*(A, B))$  and  $(G, H) \cap (A, B) \subseteq b\text{-int}(G, H) \cap b\text{-int}(b\text{-cl}^*(A, B)) = b\text{-int}((G, H) \cap b\text{-cl}^*(A, B)) \subseteq b\text{-int}(b\text{-cl}^*((G, H) \cap (A, B)))$  by Lemma 2.26. This shows that  $(G, H) \cap (A, B)$  is pre- $bl$ -open.

**Proposition 2.28** The intersection of a semi- $bl$ -open set and a  $b$ -open set is semi- $bl$ -open.

Proof. Let  $(A, B)$  be semi- $bl$ -open and  $(G, H)$  be  $b$ -open in  $\mathcal{B}^{\mathcal{M}}$ . Then  $(A, B) \subseteq b\text{-cl}^*(b\text{-int}(A, B))$  and  $b\text{-int}(G, H) = (G, H)$ .  $(G, H) \cap (A, B) \subseteq (G, H) \cap b\text{-cl}^*(b\text{-int}(A, B)) \subseteq b\text{-cl}^*((G, H) \cap b\text{-int}(A, B)) = b\text{-cl}^*(b\text{-int}(G, H) \cap b\text{-int}(A, B)) = b\text{-cl}^*(b\text{-int}((G, H) \cap (A, B)))$  by Lemma 2.26. Hence  $(A, B)$  is semi- $bl$ -open.

**Proposition 2.29** The intersection of a  $\alpha$ - $bl$ -open set and  $b$ -open set is  $\alpha$ - $bl$ -open.

Proof. Let  $(G, H)$  be a  $b$ -open and  $(A, B)$  be an  $\alpha$ - $bl$ -open in a space  $(\mathcal{B}^{\mathcal{M}}, \mathcal{I})$ . Then  $(A, B)$  is both pre- $bl$ -open and semi- $bl$ -open by (2) and (3) of Theorem 2.2.  $(A, B) \cap (G, H)$  is both pre- $bl$ -open and semi- $bl$ -open by Proposition 2.27 and 2.28. Hence by Theorem 2.9,  $(A, B) \cap (G, H)$  is  $\alpha$ - $bl$ -open.

**Proposition 2.30** The intersection of a  $b\text{-}I_b$ -open set and a  $b$ -open set is  $b\text{-}I_b$ -open.

Proof. Let  $(A, B)$  be  $b\text{-}I_b$ -open and  $(G, H)$  is  $b$ -open. Then  $(A, B) \subseteq b\text{-int}(b\text{-cl}^*(A, B)) \cup b\text{-cl}^*(b\text{-int}(A, B))$  and  $(G, H) \cap (A, B) \subseteq (G, H) \cap [b\text{-int}(b\text{-cl}^*(A, B)) \cup b\text{-cl}^*(b\text{-int}(A, B))] = [(G, H) \cap b\text{-int}(b\text{-cl}^*(A, B))] \cup [(G, H) \cap b\text{-cl}^*(b\text{-int}(A, B))] = [b\text{-int}(G, H) \cap b\text{-int}(b\text{-cl}^*(A, B))] \cup [(G, H) \cap b\text{-cl}^*(b\text{-int}(A, B))] \subseteq [b\text{-int}((G, H) \cap b\text{-cl}^*(A, B))] \cup [b\text{-cl}^*((G, H) \cap b\text{-int}(A, B))]$  by Lemma 2.26. Thus  $(G, H) \cap (A, B) \subseteq [b\text{-int}(b\text{-cl}^*((G, H) \cap (A, B)))] \cup [b\text{-cl}^*(b\text{-int}((G, H) \cap (A, B)))]$ . This shows that  $(G, H) \cap (A, B)$  is  $b\text{-}I_b$ -open.

**Proposition 2.31** The intersection of a  $\beta$ - $bl$ -open set and  $b$ -open set is  $\beta$ - $bl$ -open.

Proof. Let  $(A, B)$  be  $\beta$ - $bl$ -open and  $(G, H)$  be  $b$ -open. Then  $(A, B) \subseteq b\text{-cl}^*(b\text{-int}(b\text{-cl}^*(A, B)))$  and  $(G, H) \cap$

$(A, B) \subseteq (G, H) \cap b-cl^*(b-int(b-cl^*(A, B))) \subseteq b-cl^*((G, H) \cap b-int(b-cl^*(A, B))) = b-cl^*(b-int(G, H)) \cap b-int(b-cl^*(A, B)) = b-cl^*(b-int((G, H) \cap b-cl^*(A, B))) \subseteq b-cl^*(b-int(b-cl^*((G, H) \cap (A, B))))$  by Lemma 2.26. This proves that  $(G, H) \cap (A, B)$  is  $\beta$ - $bl$ -open.

**Remark 2.32** The intersection of two semi- $bl$ -open (resp. pre- $bl$ -open,  $b-I_b$ -open,  $\beta$ - $bl$ -open) sets need not be semi- $bl$ -open (resp. pre- $bl$ -open,  $b-I_b$ -open,  $\beta$ - $bl$ -open) as shown in the following Examples.

**Example 2.33** In Example 2.8, then

1.  $(G, H) = (\{a, b\}, \phi)$  and  $(J, K) = (\{a, c\}, \{1\})$  are semi- $bl$ -open. But  $(G, H) \cap (J, K) = (\{a\}, \phi)$  is not semi- $bl$ -open.
2.  $(G, H) = (\{b, c\}, Y)$  and  $(J, K) = (X, \{2\})$  are pre- $bl$ -open. But  $(G, H) \cap (J, K) = (\{b, c\}, \{2\})$  is not pre- $bl$ -open.
3.  $(G, H) = (\{a, c\}, \{1\})$  and  $(J, K) = (\{b, c\}, \{2\})$  are  $b-I_b$ -open. But  $(G, H) \cap (J, K) = (\{c\}, \phi)$  is not  $b-I_b$ -open.
4.  $(G, H) = (\{b, c\}, \{1\})$  and  $(J, K) = (\{c\}, \{2\})$  are  $\beta$ - $bl$ -open. But  $(G, H) \cap (J, K) = (\{c\}, \phi)$  is not  $\beta$ - $bl$ -open.

**Remark 2.34** Obviously, arbitrary intersection of  $b$ -open sets (resp. pre- $bl$ -open,  $b-I_b$ -open,  $\beta$ - $bl$ -open) need not be again a semi- $bl$ -open (resp. pre- $bl$ -open,  $b-I_b$ -open,  $\beta$ - $bl$ -open) set.

**Remark 2.35** Finite union of semi- $bl$ -open sets (resp.  $\alpha$ - $bl$ -open, pre- $bl$ -open,  $b-I_b$ -open,  $\beta$ - $bl$ -open) is again a semi- $bl$ -open (resp.  $\alpha$ - $bl$ -open, pre- $bl$ -open,  $b-I_b$ -open,  $\beta$ - $bl$ -open) set.

Proof. If  $(A, B)$  and  $(C, D)$  are semi- $bl$ -open then  $(A, B) \subseteq b-cl^*(b-int(A, B)) \subseteq b-cl^*(b-int((A, B) \cup (C, D)))$  and  $(C, D) \subseteq b-cl^*(b-int(C, D)) \subseteq b-cl^*(b-int((A, B) \cup (C, D)))$ . Therefore  $((A, B) \cup (C, D)) \subseteq b-cl^*(b-int(A, B) \cup (C, D))$ . Thus  $(A, B) \cup (C, D)$  is semi- $bl$ -open. Similarly other results.

**Remark 2.36** Obviously, arbitrary union of semi- $bl$ -open sets (resp.  $\alpha$ - $bl$ -open, pre- $bl$ -open,  $b-I_b$ -open,  $\beta$ - $bl$ -open) is again a semi- $bl$ -open (resp.  $\alpha$ - $bl$ -open, pre- $bl$ -open,  $b-I_b$ -open,  $\beta$ - $bl$ -open) set as shown in the following example.

**Example 2.37** Let  $X = \{a, b\}$ ,  $(Y = \{1, 2\})$  and  $\mathcal{M} = \{(\phi, \phi), (\{a\}, \phi), (X, Y)\}$ . Let the ideal be  $\mathcal{I} = \{(\phi, \phi)\}$ . Then

1.  $(G, H) = (\{a\}, \{1\})$  and  $(J, K) = (\{a\}, \{2\})$  are semi- $bl$ -open. Then  $(G, H) \cup (J, K) = (\{a\}, Y)$  is semi- $bl$ -open.
2.  $(G, H) = (\{a\}, \{2\})$  and  $(J, K) = (\{b\}, \{2\})$  are pre- $bl$ -open. Then  $(G, H) \cup (J, K) = (X, \{2\})$  is pre- $bl$ -open.
3.  $(G, H) = (\phi, \{1\})$  and  $(J, K) = (\{b\}, \{2\})$  are  $b-I_b$ -open. Then  $(G, H) \cup (J, K) = (\{b\}, Y)$  is  $b-I_b$ -open.
4.  $(G, H) = (\{b\}, \phi)$  and  $(J, K) = (\{a\}, \{1\})$  are  $\beta$ - $bl$ -open. Then  $(G, H) \cup (J, K) = (X, \{1\})$  is  $\beta$ - $bl$ -open.

### 3 Conclusion

The weak forms of binary open sets in ideal binary topological spaces, known as  $\alpha$ - $bl$ -open sets, semi- $bl$ -open sets, pre- $bl$ -open sets,  $b-I_b$ -open sets, and  $\beta$ - $bl$ -open sets, were introduced and examined in this study. Additionally, we described the connections between them and the associated characteristics. Furthermore,  $b$ -open sets and  $bl$ -open sets are shown to be independent.

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