



Existence and The Structure of The Solution Set for Hemivariational Inequalities Under Hybrid-Kind Monotonicity on Unbounded Domains

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Abstract

We study the hemivariational inequality $HVI(\beta\eta\xi)$ under $(\eta\xi\sigma\psi - p)$ -monotonicity, on closed convex subsets K in a reflexive Banach space. First, we give a concrete coercivity condition and use it to prove existence of a solution when K is unbounded. The proof truncates K into bounded balls, solves the problem on each ball, and shows the solutions stay away from the boundary once the ball is large enough. Second, we study the solution set S . We show every solution lies in one fixed ball, so S is bounded. We show S is weakly closed, hence weakly compact. Third, under a strong monotonicity hypothesis, S is a single point, and compactness is then trivial. We show why norm-compactness of S does not follow from our hypotheses when S has more than one point, and we record this as an open problem.

2020 Mathematics Subject Classification: 49J40, 47H04, 47J20, 46B10.

Keywords: Hemivariational inequalities; Unbounded domains; Weak compactness; Truncation method; Clarke derivative.

1 Introduction

Hemivariational inequalities model problems with a nonsmooth, nonconvex energy term. The tools used in this paper were developed in the following order.

1961 – Fan proved a KKM-type fixed-point lemma [1]. It is the existence tool used throughout this paper.

1968 – Kuratowski introduced the measure of non-compactness used in Section 5 [2].

1981 – Clarke defined the generalized gradient of a locally Lipschitz functional [3]. It underlies the term J^0 in $HVI(\beta\eta\xi)$

1990 – Zeidler's monograph collects the monotone-operator theory this paper builds on [4].

2009 – Lee, Khan, and Salahuddin introduced hybrid-kind operators and ϖ -coercivity [5]. We use their coercivity notion in Definition 2.6.

2010 – Costea and Lupu proved existence for monotone variational-hemivariational inequalities on bounded K via the Fan-KKM technique, and gave sufficient conditions for existence on unbounded K by truncation [6]. Theorems 2.4 and 4.1 below adapt this technique to the hybrid-kind setting.

2011 – Brézis's textbook supplies the reflexive-space facts used in Section 5, such as Kakutani's theorem [7].

2012 – Mahato and Nahak gave weak-coercivity conditions for equilibrium problems on unbounded sets [8]. Condition 3.1 follows the same spirit.

2016 – Hashoosh, Alimohammady, and Kalleji studied equilibrium problems with α -monotone bifunctions [9].

2017 – Hashoosh proved existence and uniqueness for a class of hemivariational inequalities [10].

2023 – Khisbag, Hashoosh, and Kalt introduced a class of invexity functions for vector variational problems on H -spaces [14].

2024 – Hashoosh, Hameed, and Mohsen introduced a new monotone-type operator for hemivariational inequalities via the KKM-mapping technique [15].

2025 – Carl treated hemivariational inequalities on unbounded cylindrical domains without any coercivity hypothesis, using different structural assumptions instead [16].

2025 – Cen, Costea, Dumitrescu, and Stanciu combined coercivity with weak compactness of the whole solution set for coupled quasi-variational-hemivariational systems [17]. Theorem 5.3 below is the analogous statement for a single hemivariational inequality.

2026 – Kozłowski gave a unified treatment of the Kadec–Klee property [18], relevant to the open problem in Section 5.

2026 – Alrikabi, Barghooth, Muazi, and Hashoosh gave a Fan–KKM existence theory for vector hemivariational-like inequalities under a two-map monotonicity condition [19], closely related in spirit to the hybrid-kind monotonicity used here.

Related equilibrium-type formulations including trifunction and hemiequilibrium versions, and well-posedness questions for them have been treated along parallel lines [9, 10, 11, 12, 13, 14, 15, 19, 20], building on the hybrid-kind operators of Lee, Khan, and Salahuddin [5] and the KKM technique of Costea and Lupu [6], both discussed above.

Still, none of this work proves existence for $HVI(\beta\eta\xi)$ when K is unbounded. None of it studies the shape of the solution set S either. This paper does both.

Unbounded K arises often in practice: a displacement field on an unbounded elastic body, a strategy set with no upper bound, a price variable in an economic model. We keep a coercivity assumption, but we write it as a plain, pointwise condition (Condition 3.1) instead of a limit. That makes it easy to check, and easy to plug straight into a truncation argument.

This paper makes three contributions.

First, existence on a bounded K . Theorem 2.4 proves it with the Fan–KKM technique, following Costea and Lupu [6, Theorem 2].

Second, existence on an unbounded K . Condition 3.1 makes the coercivity assumption concrete, in the spirit of condition (H3) of Costea and Lupu [6, Theorem 3] and of the weak-coercivity conditions of Mahato and Nahak [8, Condition (iv), Theorem 3.3; Condition (v), Theorem 4.9]. Theorem 4.1 then proves existence. The proof cuts K into balls $K_r = K \cap \overline{B}(0, r)$, solves the problem on each ball with Theorem 2.4, and shows the solutions stay away from the boundary once r is large enough. This is the same truncation idea as [6, Theorem 3], carried over here to the penalty term ξ and the Clarke derivative J^0 .

Third, the shape of the solution set S . Proposition 5.1 shows every solution sits inside one fixed ball. Theorem 5.3 shows S is weakly closed, so it is weakly compact. Theorem 2.7 gives a strong-monotonicity condition under which S is a single point (Corollary 5.5); norm-compactness is then trivial. Without that condition, we show why norm-compactness does not follow automatically, and we leave this as an open problem (Open Problem 5.7).

The paper is self-contained. Section 2 recalls only classical, published facts: Clarke’s calculus [3], the Fan–KKM lemma [1], the Kuratowski measure [2]. Everything else the equivalence theorem, the bounded-domain existence theorem, the uniqueness theorem is proved here from those facts.

2 Preliminaries

Throughout, B is a real reflexive Banach space, with dual B^* and duality pairing $\langle \cdot, \cdot \rangle$. The set $K \subset B$ is nonempty, closed, and convex. Unless we say otherwise, K is unbounded.

Definition 2.1. An element $\theta \in K$ solves $HVI(\beta\eta\xi)$ if $\langle \beta(\theta, u), \eta(\zeta, \theta) \rangle + \xi(\zeta) - \xi(\theta) + J^0(\theta, \zeta - \theta) \geq 0 \quad \forall \zeta \in K, \forall u \in T(\theta)$. We write $S \subset K$ for the set of all such θ . We always assume $T(\theta) \neq \emptyset$ for every $\theta \in K$.

Throughout, $J: B \rightarrow \mathbb{R}$ denotes a locally Lipschitz functional, and $J^0(\theta, d)$ denotes its Clarke generalized directional derivative at $\theta \in B$ in the direction $d \in B$,

$$J^0(\theta, d) := \limsup_{\substack{w \rightarrow \theta, t \downarrow 0}} \frac{J(w+td) - J(w)}{t}.$$

We recall from Clarke [3] that $d \mapsto J^0(\theta, d)$ is finite, positively homogeneous ($J^0(\theta, td) = tJ^0(\theta, d)$ for $t > 0$, and $J^0(\theta, 0) = 0$), and subadditive ($J^0(\theta, d_1 + d_2) \leq J^0(\theta, d_1) + J^0(\theta, d_2)$). It is also upper semicontinuous in (θ, d) . We use these three facts often below.

Subadditivity only holds at one fixed point θ . For example it gives $J^0(\theta, d) + J^0(\theta, -d) \geq J^0(\theta, 0) = 0$. But it says nothing about $J^0(\theta_1, d) + J^0(\theta_2, -d)$ when $\theta_1 \neq \theta_2$. This second, mixed kind of expression shows up in the proof of Theorem 2.7. Condition $(\dagger)$ there absorbs it directly, instead of relying on a two-point subadditivity that does not hold.

We also recall the following continuity notion for a set-valued map $T:K \rightarrow 2^{B^*}$ with respect to a direction map $\eta:K \times K \rightarrow B$ and the hybrid-kind pairing $\beta:K \times B^* \rightarrow B^*$: T is η -hemicontinuous with respect to β if for any $\theta, \varsigma \in K$, writing $\theta_t := \theta + t(\varsigma - \theta) \in K$ for $t \in (0,1]$, and for every $u \in T(\theta)$, there exists a selection $u_t \in T(\theta_t)$, $t \in (0,1]$, such that

$$\lim_{t \rightarrow 0^+} \langle \beta(\theta, u_t), \eta(\varsigma, \theta) \rangle = \langle \beta(\theta, u), \eta(\varsigma, \theta) \rangle.$$

Here $\beta(\theta, \cdot)$ is always evaluated at the fixed base point θ , never at the moving point θ_t . This matches how β is used everywhere else in this paper: its first argument is always the candidate solution point θ (see Definition 2.2, the inequality (*), and Theorem 2.7). This is also the direction the proofs below actually need. We start from an arbitrary $u \in T(\theta)$ the same quantifier used in the definition of a solution and we build an approximating selection along the segment. We do not just bound a supremum over $T(\theta)$ that might never be attained. This mirrors the lower-hemicontinuity condition of Costea and Lupu [6, Definition 1(ii)-(iii)], which also starts from a fixed, arbitrary element of the value set at the limit point. It reduces to their condition when β is the identity pairing $\beta(\theta, u) = u$.

Definition 2.2. Let $T:K \rightarrow 2^{B^*}$, $\beta:K \times B^* \rightarrow B^*$, $\eta:K \times K \rightarrow B$, $\xi:K \rightarrow \mathbb{R} \cup \{+\infty\}$ with $K \cap \text{dom}(\xi) \neq \emptyset$ (where $\text{dom}(\xi) = \{x \in B: \xi(x) < +\infty\}$), $\sigma: B \rightarrow \mathbb{R}$ satisfying $\sigma(tz) = \pi(t)\sigma(z)$ for all $z \in B$, $t > 0$, for some $\pi: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ with $\lim_{t \rightarrow 0^+} \pi(t)/t = 0$, and $\psi: K \times K \rightarrow \mathbb{R}$ satisfying $\lim_{t \rightarrow 0^+} \left[\frac{\pi(t)\sigma(\varsigma - \theta)}{t} + \frac{\psi(\theta, t\varsigma + (1-t)\theta)}{t} \right] = 0 \quad \forall \theta, \varsigma \in K$. T is $(\eta\xi\sigma\psi - p)$ -monotone with respect to β if, for all $\theta, \varsigma \in K$ and every $u \in T(\theta)$, $\langle \beta(\theta, u), \eta(\varsigma, \theta) \rangle + \xi(\varsigma) - \xi(\theta) + J^0(\theta, \varsigma - \theta) \geq 0$ implies $\langle \beta(\theta, v), \eta(\varsigma, \theta) \rangle + \xi(\varsigma) - \xi(\theta) + J^0(\theta, \varsigma - \theta) \geq \sigma(\varsigma - \theta) + \psi(\theta, \varsigma) \quad \forall v \in T(\varsigma)$. That is, the implication is required for each individual $u \in T(\theta)$ on its own.

Theorem 2.3. Suppose K is nonempty, closed, and convex, T is η -hemicontinuous and $(\eta\xi\sigma\psi - p)$ -monotone with respect to β , $\eta(\theta, \theta) = 0$ for all $\theta \in K$, η is affine in its first argument, $J: B \rightarrow \mathbb{R}$ is locally Lipschitz, and ξ is convex and lower semicontinuous on K . Then $\theta \in K$ solves $HVI(\beta\eta\xi)$ if and only if

$$\langle \beta(\theta, v), \eta(\varsigma, \theta) \rangle + \xi(\varsigma) - \xi(\theta) + J^0(\theta, \varsigma - \theta) \geq \sigma(\varsigma - \theta) + \psi(\theta, \varsigma) \quad \forall \varsigma \in K, v \in T(\varsigma). \quad (*)$$

Proof. ($\Rightarrow$) Suppose θ solves $HVI(\beta\eta\xi)$. Then for all $\varsigma \in K$ and $u \in T(\theta)$,

$$\langle \beta(\theta, u), \eta(\varsigma, \theta) \rangle + \xi(\varsigma) - \xi(\theta) + J^0(\theta, \varsigma - \theta) \geq 0.$$

Since T is $(\eta\xi\sigma\psi - p)$ -monotone, this implies, for all $v \in T(\varsigma)$, exactly (*). So θ solves (*).

($\Leftarrow$) Conversely, suppose $\theta \in K$ satisfies (*). Fix $\varsigma \in K$ and an arbitrary $u \in T(\theta)$; we must show $\langle \beta(\theta, u), \eta(\varsigma, \theta) \rangle + \xi(\varsigma) - \xi(\theta) + J^0(\theta, \varsigma - \theta) \geq 0$. For $t \in (0,1]$, set $\theta_t := t\varsigma + (1-t)\theta \in K$ (by convexity of K). By η -hemicontinuity of T with respect to β , there is a selection $u_t \in T(\theta_t)$, $t \in (0,1]$, with

$$\lim_{t \rightarrow 0^+} \langle \beta(\theta, u_t), \eta(\varsigma, \theta) \rangle = \langle \beta(\theta, u), \eta(\varsigma, \theta) \rangle. \quad (2.1)$$

Applying (*) at the pair (θ, θ_t) with this $u_t \in T(\theta_t)$,

$$\langle \beta(\theta, u_t), \eta(\theta_t, \theta) \rangle + \xi(\theta_t) - \xi(\theta) + J^0(\theta, \theta_t - \theta) \geq \sigma(\theta_t - \theta) + \psi(\theta, \theta_t). \quad (2.2)$$

Since $\theta_t - \theta = t(\varsigma - \theta)$, the defining properties of σ give $\sigma(\theta_t - \theta) = \pi(t)\sigma(\varsigma - \theta)$. By hypothesis, $\eta(\theta, \theta) = 0$ and η is affine in its first argument, so

$$\eta(\theta_t, \theta) = \eta(t\varsigma + (1-t)\theta, \theta) = t\eta(\varsigma, \theta) + (1-t)\eta(\theta, \theta) = t\eta(\varsigma, \theta).$$

By convexity of ξ , $\xi(\theta_t) \leq t\xi(\varsigma) + (1-t)\xi(\theta)$, i.e. $\xi(\theta_t) - \xi(\theta) \leq t(\xi(\varsigma) - \xi(\theta))$. By positive homogeneity of J^0 (recalled above), $J^0(\theta, \theta_t - \theta) = tJ^0(\theta, \varsigma - \theta)$. Substituting these three facts into (2.2),

$$\begin{aligned} t[\langle \beta(\theta, u_t), \eta(\varsigma, \theta) \rangle + \xi(\varsigma) - \xi(\theta) + J^0(\theta, \varsigma - \theta)] \\ \geq \langle \beta(\theta, u_t), \eta(\theta_t, \theta) \rangle + \xi(\theta_t) - \xi(\theta) + J^0(\theta, \theta_t - \theta) \\ \geq \pi(t)\sigma(\varsigma - \theta) + \psi(\theta, \theta_t), \end{aligned}$$

where the first inequality uses $\xi(\theta_t) - \xi(\theta) \leq t(\xi(\varsigma) - \xi(\theta))$. Dividing by $t > 0$,

$$\langle \beta(\theta, u_t), \eta(\varsigma, \theta) \rangle + \xi(\varsigma) - \xi(\theta) + J^0(\theta, \varsigma - \theta) \geq \frac{\pi(t)}{t}\sigma(\varsigma - \theta) + \frac{\psi(\theta, \theta_t)}{t}. \quad (2.3)$$

By the defining property of ψ , the right side of (2.3) tends to 0 as $t \rightarrow 0^+$. By (2.1), the first term on the left tends to $\langle \beta(\theta, u), \eta(\varsigma, \theta) \rangle$. The quantity $\xi(\varsigma) - \xi(\theta) + J^0(\theta, \varsigma - \theta)$ does not depend on t . So letting $t \rightarrow 0^+$ in (2.3) along the sequence u_t gives an honest limit, not merely a limsup:

$$\langle \beta(\theta, u), \eta(\varsigma, \theta) \rangle + \xi(\varsigma) - \xi(\theta) + J^0(\theta, \varsigma - \theta) \geq 0.$$

Since $u \in T(\theta)$ and $\varsigma \in K$ were both arbitrary, θ solves $HVI(\beta\eta\xi)$.

This argument is the hybrid-kind generalization with the general pairing β , the general affine direction map η , and the penalty term ξ carried through the convexity estimate of the segment argument used by Costea and Lupu to prove $S^* = S$ under lower hemicontinuity [6, proof of Theorem 1]. $\square$

We will use Theorem 2.4 repeatedly in the next section.

Theorem 2.4. Suppose K is nonempty, bounded, closed, and convex, and $T:K \rightarrow 2^{B^*}$ is η -hemicontinuous and $(\eta\xi\sigma\psi - p)$ -monotone with respect to β . Assume:

1. $\eta(\theta, \theta) = 0$ for all $\theta \in K$;
2. η is affine in its first argument;

3. $\sigma: B \rightarrow \mathbb{R}$ and $\psi(\cdot, \theta)$ are lower semicontinuous for all $\theta \in K$;
4. $\xi: K \rightarrow \mathbb{R} \cup \{+\infty\}$ is convex and lower semicontinuous;
5. $J: B \rightarrow \mathbb{R}$ is locally Lipschitz;
6. for each fixed $\varsigma \in K$ and $v \in T(\varsigma)$, the map $\theta \mapsto \langle \beta(\theta, v), \eta(\varsigma, \theta) \rangle$ is weakly sequentially upper semicontinuous on K .

Then $HVI(\beta\eta\xi)$ has at least one solution $\theta \in K$.

Proof. We adapt the Fan-KKM argument of Costea and Lupu [6, proof of Theorem 2] to the hybrid-kind setting. Define $F, G: K \rightarrow 2^K$ by

$$F(\varsigma) := \{\theta \in K: \langle \beta(\theta, u), \eta(\varsigma, \theta) \rangle + \xi(\varsigma) - \xi(\theta) + J^0(\theta, \varsigma - \theta) \geq 0 \ \exists u \in T(\theta)\},$$

$$G(\varsigma) := \{\theta \in K: \langle \beta(\theta, v), \eta(\varsigma, \theta) \rangle + \xi(\varsigma) - \xi(\theta) + J^0(\theta, \varsigma - \theta) \geq \sigma(\varsigma - \theta) + \psi(\theta, \varsigma) \ \forall v \in T(\varsigma)\}.$$

Step 1: F is a KKM-mapping. Suppose not. Then there is a finite set $\{\varsigma_1, \dots, \varsigma_n\} \subset K$ and $\varsigma_0 = \sum_{i=1}^n t_i \varsigma_i \in \text{co}\{\varsigma_1, \dots, \varsigma_n\}$ (with $t_i \geq 0, \sum_i t_i = 1$) such that $\varsigma_0 \notin \bigcup_{i=1}^n F(\varsigma_i)$. By definition of F , for every $u_0 \in T(\varsigma_0)$ and every $i=1, \dots, n$,

$$\langle \beta(\varsigma_0, u_0), \eta(\varsigma_i, \varsigma_0) \rangle + \xi(\varsigma_i) - \xi(\varsigma_0) + J^0(\varsigma_0, \varsigma_i - \varsigma_0) < 0.$$

Multiplying by $t_i \geq 0$ and summing over i : by hypotheses (1)-(2), $\sum_i t_i \eta(\varsigma_i, \varsigma_0) = \eta(\varsigma_0, \varsigma_0) = 0$; by convexity of ξ (hypothesis (4)), $\sum_i t_i \xi(\varsigma_i) \geq \xi(\varsigma_0)$; and by subadditivity and positive homogeneity of J^0 , $\sum_i t_i J^0(\varsigma_0, \varsigma_i - \varsigma_0) \geq J^0(\varsigma_0, \sum_i t_i (\varsigma_i - \varsigma_0)) = J^0(\varsigma_0, 0) = 0$. Adding these three facts gives

$$0 > \sum_{i=1}^n t_i [\langle \beta(\varsigma_0, u_0), \eta(\varsigma_i, \varsigma_0) \rangle + \xi(\varsigma_i) - \xi(\varsigma_0) + J^0(\varsigma_0, \varsigma_i - \varsigma_0)] \geq \langle \beta(\varsigma_0, u_0), 0 \rangle + 0 + 0 = 0,$$

a contradiction. Hence F is a KKM-mapping.

Step 2: $F(\varsigma) \subset G(\varsigma)$ for all $\varsigma \in K$. If $\theta \in F(\varsigma)$, then there is a specific $u \in T(\theta)$ with $\langle \beta(\theta, u), \eta(\varsigma, \theta) \rangle + \xi(\varsigma) - \xi(\theta) + J^0(\theta, \varsigma - \theta) \geq 0$. By Definition 2.2, applied to this same u , this implies, for all $v \in T(\varsigma)$, $\langle \beta(\theta, v), \eta(\varsigma, \theta) \rangle + \xi(\varsigma) - \xi(\theta) + J^0(\theta, \varsigma - \theta) \geq \sigma(\varsigma - \theta) + \psi(\theta, \varsigma)$, i.e. $\theta \in G(\varsigma)$. In particular, since F is a KKM-mapping, so is G .

Step 3: $G(\varsigma)$ is weakly closed, for every $\varsigma \in K$. Let $\{\theta_n\} \subset G(\varsigma)$ with $\theta_n \rightarrow \theta_0$ weakly in B . For each n and each $v \in T(\varsigma)$,

$$\langle \beta(\theta_n, v), \eta(\varsigma, \theta_n) \rangle + \xi(\varsigma) - \xi(\theta_n) + J^0(\theta_n, \varsigma - \theta_n) \geq \sigma(\varsigma - \theta_n) + \psi(\theta_n, \varsigma).$$

By hypothesis (6), $\limsup_n \langle \beta(\theta_n, v), \eta(\varsigma, \theta_n) \rangle \leq \langle \beta(\theta_0, v), \eta(\varsigma, \theta_0) \rangle$; by hypothesis (4), $\liminf_n \xi(\theta_n) \geq \xi(\theta_0)$; by upper semicontinuity of J^0 , $\limsup_n J^0(\theta_n, \varsigma - \theta_n) \leq J^0(\theta_0, \varsigma - \theta_0)$; and by hypothesis (3), $\liminf_n [\sigma(\varsigma - \theta_n) + \psi(\theta_n, \varsigma)] \geq \sigma(\varsigma - \theta_0) + \psi(\theta_0, \varsigma)$. Taking $\limsup_n$ in the displayed inequality and combining these four facts,

$$\langle \beta(\theta_0, v), \eta(\varsigma, \theta_0) \rangle + \xi(\varsigma) - \xi(\theta_0) + J^0(\theta_0, \varsigma - \theta_0) \geq \sigma(\varsigma - \theta_0) + \psi(\theta_0, \varsigma),$$

so $\theta_0 \in G(\varsigma)$. Since K is bounded, closed, and convex in the reflexive space B , K is weakly compact, so each $G(\varsigma) \subset K$ is also weakly compact.

Step 4: apply the Fan-KKM lemma. Since G is a KKM-mapping with weakly compact values, the Fan-KKM lemma [1] (applied in the weak topology on K) gives $\theta^* \in \bigcap_{\varsigma \in K} G(\varsigma)$, i.e. θ^* satisfies (*) of Theorem 2.3 for every $\varsigma \in K$. By that theorem (whose hypotheses (1),(2),(4),(5) are exactly hypotheses (1),(2),(4),(5) above, together with η -hemicontinuity of T), θ^* solves $HVI(\beta\eta\xi)$. $\square$

We record separately, for use in Section 5, the fact established in Step 3 above.

Lemma 2.5. Under hypotheses (3), (4), (6) of Theorem 2.4 and upper semicontinuity of J^0 , the set $G(\varsigma) := \{\theta \in K: \langle \beta(\theta, v), \eta(\varsigma, \theta) \rangle + \xi(\varsigma) - \xi(\theta) + J^0(\theta, \varsigma - \theta) \geq \sigma(\varsigma - \theta) + \psi(\theta, \varsigma) \ \forall v \in T(\varsigma)\}$ is weakly (sequentially) closed in K , for every $\varsigma \in K$ (with K not necessarily bounded: the argument in Step 3 above uses only the semicontinuity hypotheses along the given weakly convergent sequence, not boundedness of K).

Definition 2.6 (Lee, Khan, and Salahuddin [5]). Let $N: K \times B^* \rightarrow B^*$, $\varpi: K \times K \rightarrow B$, and $L: K \times K \rightarrow \mathbb{R} \cup \{+\infty\}$. T is ϖ -coercive with respect to L if there exists $\theta_0 \in K$ such that, for all $u \in T(\theta)$ and $u_0 \in T(\theta_0)$,
$$\frac{\langle N(\theta, u) - N(\theta_0, u_0), \varpi(\theta, \theta_0) \rangle + L(\theta, \theta_0)}{\|\varpi(\theta_0, \theta)\|} \rightarrow +\infty \quad \text{whenever} \quad \|\theta\| \rightarrow \infty, \theta \in K.$$

Theorem 2.7. Suppose, in addition to the hypotheses of Theorem 2.3, that there is $c > 0$ with
$$\sigma(\varsigma - \theta) + \psi(\theta, \varsigma) + \sigma(\theta - \varsigma) + \psi(\varsigma, \theta) - J^0(\theta, \varsigma - \theta) - J^0(\varsigma, \theta - \varsigma) \geq 2c \|\varsigma - \theta\|^2 \quad \forall \theta, \varsigma \in K, \quad (\dagger)$$
 and that β satisfies $\langle \beta(\theta_1, v_2), \eta(\theta_2, \theta_1) \rangle + \langle \beta(\theta_2, v_1), \eta(\theta_1, \theta_2) \rangle \leq 0 \quad \forall \theta_1, \theta_2 \in K, v_1 \in T(\theta_1), v_2 \in T(\theta_2). \quad (\ddagger)$ Then $HVI(\beta\eta\xi)$ has at most one solution.

Proof. Suppose $\theta_1, \theta_2 \in K$ both solve $HVI(\beta\eta\xi)$. Applying $(\eta\xi\sigma\psi - p)$ -monotonicity with $\theta = \theta_1, \varsigma = \theta_2$ (using that θ_1 solves $HVI(\beta\eta\xi)$), for all $v_2 \in T(\theta_2)$,

$$\langle \beta(\theta_1, v_2), \eta(\theta_2, \theta_1) \rangle + \xi(\theta_2) - \xi(\theta_1) + J^0(\theta_1, \theta_2 - \theta_1) \geq \sigma(\theta_2 - \theta_1) + \psi(\theta_1, \theta_2). \quad (2.4)$$

Symmetrically, applying $(\eta\xi\sigma\psi - p)$ -monotonicity with $\theta = \theta_2, \varsigma = \theta_1$, for all $v_1 \in T(\theta_1)$,

$$\langle \beta(\theta_2, v_1), \eta(\theta_1, \theta_2) \rangle + \xi(\theta_1) - \xi(\theta_2) + J^0(\theta_2, \theta_1 - \theta_2) \geq \sigma(\theta_1 - \theta_2) + \psi(\theta_2, \theta_1). \quad (2.5)$$

Adding (2.4) and (2.5), the ξ -terms cancel:

$$\begin{aligned} & \langle \beta(\theta_1, v_2), \eta(\theta_2, \theta_1) \rangle + \langle \beta(\theta_2, v_1), \eta(\theta_1, \theta_2) \rangle + J^0(\theta_1, \theta_2 - \theta_1) + J^0(\theta_2, \theta_1 - \theta_2) \\ & \geq \sigma(\theta_2 - \theta_1) + \psi(\theta_1, \theta_2) + \sigma(\theta_1 - \theta_2) + \psi(\theta_2, \theta_1). \end{aligned}$$

By (†), applied at $\theta = \theta_1, \zeta = \theta_2$, the right side satisfies

$$\begin{aligned} & \sigma(\theta_2 - \theta_1) + \psi(\theta_1, \theta_2) + \sigma(\theta_1 - \theta_2) + \psi(\theta_2, \theta_1) \\ & \geq 2c \|\theta_2 - \theta_1\|^2 + J^0(\theta_1, \theta_2 - \theta_1) + J^0(\theta_2, \theta_1 - \theta_2). \end{aligned}$$

Combining the last two displays,

$$\begin{aligned} & \langle \beta(\theta_1, v_2), \eta(\theta_2, \theta_1) \rangle + \langle \beta(\theta_2, v_1), \eta(\theta_1, \theta_2) \rangle + J^0(\theta_1, \theta_2 - \theta_1) + J^0(\theta_2, \theta_1 - \theta_2) \\ & \geq 2c \|\theta_2 - \theta_1\|^2 + J^0(\theta_1, \theta_2 - \theta_1) + J^0(\theta_2, \theta_1 - \theta_2). \end{aligned}$$

The two J^0 -terms appear identically on both sides and cancel no claim about their sign, individually or combined, is needed leaving

$$\langle \beta(\theta_1, v_2), \eta(\theta_2, \theta_1) \rangle + \langle \beta(\theta_2, v_1), \eta(\theta_1, \theta_2) \rangle \geq 2c \|\theta_2 - \theta_1\|^2.$$

By (‡), the left side is ≤ 0 . So $0 \geq 2c \|\theta_2 - \theta_1\|^2$, and since $c > 0$, this forces $\theta_2 = \theta_1$. $\square$

3 A Concrete Coercivity Condition

Definition 2.6 is very general. It is hard to use directly in an existence proof. As stated, it is a limit as $\|\theta\| \rightarrow \infty$. So it only bounds a sequence of solutions that already exists it cannot produce one. We now write down a concrete condition that we can use directly. It is in the spirit of condition (H3) of Costea and Lupu [6, Theorem 3] and of the weak-coercivity conditions of Mahato and Nahak [8, Condition (iv), Theorem 3.3; Condition (v), Theorem 4.9], adapted here to ξ and J^0 .

Condition 3.1. There exists $\theta_0 \in K$ and $R_0 > \|\theta_0\|$ such that

$$\langle \beta(\theta, u), \eta(\theta_0, \theta) \rangle + \xi(\theta_0) - \xi(\theta) + J^0(\theta, \theta_0 - \theta) < 0 \quad (3.1)$$

for every $\theta \in K$ with $\|\theta\| \geq R_0$ and every $u \in T(\theta)$.

Remark 3.2. Condition 3.1 carries the same idea as Definition 2.6: a fixed point θ_0 eventually dominates every far-away $\theta \in K$. We do not force an exact algebraic match between the two. That would need L, N, ϖ pinned down as precisely as the original definition, and nothing below needs that. Take the classical variational inequality case: $\eta(\zeta, \theta) = \zeta - \theta$, $\xi \equiv 0$, $J \equiv 0$. Then Condition 3.1 reduces to $\langle \beta(\theta, u), \theta_0 - \theta \rangle < 0$ for $\|\theta\|$ large. This is the standard coercivity used for monotone variational inequalities on unbounded sets (see, e.g., [4]). It matches condition (iv) of [8, Theorem 3.3] and condition (v) of [8, Theorem 4.9] in their special cases $\xi \equiv 0, J \equiv 0$.

Proposition 3.3. Let $\theta_0 \in K$. Suppose the objects N, ϖ, L of Definition 2.6 match those of $HVI(\beta\eta\xi)$ as follows:

1. $N = \beta$, $\varpi(\theta_0, \theta) = \eta(\theta_0, \theta)$, and $\varpi(\theta, \theta_0) = -\eta(\theta_0, \theta)$ for all $\theta \in K$;
2. $L(\theta, \theta_0) = \xi(\theta) - \xi(\theta_0) - J^0(\theta, \theta_0 - \theta)$ for all $\theta \in K$;
3. $\eta(\theta_0, \theta) \neq 0$ whenever $\theta \neq \theta_0$;
4. $M := \sup\{\|\beta(\theta_0, u_0)\|_{B^*} : u_0 \in T(\theta_0)\} < \infty$.

If T is ϖ -coercive with respect to L at this θ_0 , then Condition 3.1 holds at the same θ_0 .

Proof. Fix $u \in T(\theta)$ and $u_0 \in T(\theta_0)$. Write

$$E(\theta, u) := \langle \beta(\theta, u), \eta(\theta_0, \theta) \rangle + \xi(\theta_0) - \xi(\theta) + J^0(\theta, \theta_0 - \theta),$$

the left-hand side of (3.1). Substituting (a) and (b) into the numerator of Definition 2.6 and expanding,

$$\langle N(\theta, u) - N(\theta_0, u_0), \varpi(\theta, \theta_0) \rangle + L(\theta, \theta_0) = -E(\theta, u) + \langle \beta(\theta_0, u_0), \eta(\theta_0, \theta) \rangle.$$

By ϖ -coercivity, dividing by $\|\varpi(\theta_0, \theta)\| = \|\eta(\theta_0, \theta)\|$ and letting $\|\theta\| \rightarrow \infty$,

$$\frac{-E(\theta, u) + \langle \beta(\theta_0, u_0), \eta(\theta_0, \theta) \rangle}{\|\eta(\theta_0, \theta)\|} \rightarrow +\infty.$$

By (d), $|\langle \beta(\theta_0, u_0), \eta(\theta_0, \theta) \rangle| \leq M \|\eta(\theta_0, \theta)\|$. So the second term of the numerator, divided by $\|\eta(\theta_0, \theta)\|$, stays between $-M$ and M , no matter what θ is. Hence $-E(\theta, u) / \|\eta(\theta_0, \theta)\| \rightarrow +\infty$ too. So there is $R_0 > 0$ with $-E(\theta, u) / \|\eta(\theta_0, \theta)\| > 0$ for every $\theta \in K$ with $\|\theta\| \geq R_0$ and every $u \in T(\theta)$. By (c), $\|\eta(\theta_0, \theta)\| > 0$, so $E(\theta, u) < 0$. This is exactly Condition 3.1. $\square$

Remark 3.4. Proposition 3.3 explains why we work with Condition 3.1 instead of Definition 2.6 directly. The two are not competing hypotheses. Condition 3.1 is just the special case of ϖ -coercivity that appears under the match (a)(d), plus the mild extra fact (d) that $\beta(\theta_0, \cdot)$ stays bounded on $T(\theta_0)$. Condition 3.1 is easier to use in Theorem 4.1: it is a plain inequality at each θ , which is exactly what Step 2 of that proof needs. Definition 2.6, by contrast, is stated as a limit. It cannot be applied directly to one truncated solution.

Remark 3.5. The truncation strategy follows Costea and Lupu [6] and Mahato and Nahak [8]: exhaust K by balls, solve on each ball, show the solutions avoid the boundary, then pass to a segment argument. The shape

of the coercivity hypothesis follows the same sources, as does the fact that the final step needs exact algebra, not a limit. Condition 3.1 extends this to $HVI(\beta\eta\xi)$, which carries two terms absent from [6, 8]: the general hybrid-kind pairing β , and the Clarke derivative J^0 of a nonsmooth potential J on the whole Banach space B , rather than an integral functional composed with a compact operator as in [6]. The algebra in Step 3 of Theorem 4.1 positive homogeneity of J^0 , convexity of ξ follows the same step in [6, Theorem 2], generalized to the hybrid-kind pairing. Proposition 3.3 links Condition 3.1 to the ϖ -coercivity of Lee, Khan, and Salahuddin [5].

4 Existence on Unbounded Domains

Theorem 4.1. Let B be a real reflexive Banach space and let $K \subset B$ be nonempty, closed, convex, and unbounded. Let $T:K \rightarrow 2^{B^*}$ be η -hemicontinuous and $(\eta\xi\sigma\psi - p)$ -monotone with respect to $\beta:K \times B^* \rightarrow B^*$. Assume:

1. $\eta(\theta, \theta) = 0$ for all $\theta \in K$.
2. η is affine in its first argument: $\eta(t\zeta + (1-t)\theta', \theta) = t\eta(\zeta, \theta) + (1-t)\eta(\theta', \theta)$ for all $\zeta, \theta', \theta \in K$, $t \in [0, 1]$.
3. $\sigma: B \rightarrow \mathbb{R}$ and $\psi(\cdot, \theta)$ are lower semicontinuous for all $\theta \in K$.
4. $\xi: K \rightarrow \mathbb{R} \cup \{+\infty\}$ is convex and lower semicontinuous.
5. $J: B \rightarrow \mathbb{R}$ is locally Lipschitz.
6. For each fixed $\zeta \in K$ and $v \in T(\zeta)$, the map $\theta \mapsto \langle \beta(\theta, v), \eta(\zeta, \theta) \rangle$ is weakly sequentially upper semicontinuous on K .
7. Condition 3.1 holds for some $\theta_0 \in K$ and $R_0 > \|\theta_0\|$.

Then $HVI(\beta\eta\xi)$ has at least one solution $\theta^* \in K$.

Proof. Step 1: Solve the problem on a bounded ball. Fix $r > R_0$ and set

$$K_r := K \cap \overline{B}(0, r) = \{\theta \in K: \|\theta\| \leq r\}.$$

Since $\theta_0 \in K$ and $\|\theta_0\| < R_0 < r$, K_r is nonempty. It is bounded, closed, and convex, since it is the intersection of two closed convex sets. Hypotheses (1)(6) above hold for all $\theta, \zeta \in K$, so they hold on $K_r \subset K$ too. For the same reason, T restricted to K_r is still η -hemicontinuous and $(\eta\xi\sigma\psi - p)$ -monotone. So Theorem 2.4, applied to K_r , gives $\theta_r \in K_r$ with

$$\langle \beta(\theta_r, u), \eta(\zeta, \theta_r) \rangle + \xi(\zeta) - \xi(\theta_r) + J^0(\theta_r, \zeta - \theta_r) \geq 0 \quad \forall \zeta \in K_r, \forall u \in T(\theta_r). \quad (4.1)$$

Step 2: For r large, θ_r is not on the boundary of K_r . We use Condition 3.1 to rule this out. Suppose, for contradiction, that there is a sequence $r_n \rightarrow \infty$, with $r_n > R_0$, such that $\|\theta_{r_n}\| = r_n$ for every n . Since $\theta_0 \in K$ and $\|\theta_0\| < R_0 < r_n$, we have $\theta_0 \in K_{r_n}$. Take $\zeta = \theta_0$ in (4.1) (with $r = r_n$). This holds for every $u \in T(\theta_{r_n})$, so let $u_n \in T(\theta_{r_n})$ be any such element. Then

$$\langle \beta(\theta_{r_n}, u_n), \eta(\theta_0, \theta_{r_n}) \rangle + \xi(\theta_0) - \xi(\theta_{r_n}) + J^0(\theta_{r_n}, \theta_0 - \theta_{r_n}) \geq 0. \quad (4.2)$$

On the other hand, $\|\theta_{r_n}\| = r_n \rightarrow \infty$, so eventually $\|\theta_{r_n}\| \geq R_0$. Condition 3.1, applied at $\theta = \theta_{r_n}$ with this same u_n , gives

$$\langle \beta(\theta_{r_n}, u_n), \eta(\theta_0, \theta_{r_n}) \rangle + \xi(\theta_0) - \xi(\theta_{r_n}) + J^0(\theta_{r_n}, \theta_0 - \theta_{r_n}) < 0. \quad (4.3)$$

Inequalities (4.2) and (4.3) contradict each other for large n . So no such sequence exists. Therefore, by contradiction, there is $\bar{r} > R_0$ such that every solution θ_r of (4.1) satisfies $\|\theta_r\| < \bar{r}$ whenever $r \geq \bar{r}$. Fix such an r and write $\theta_r^* := \theta_r$, so $\|\theta_r^*\| < r$.

Step 3: θ_r^* solves the problem on all of K . The truncated inequality (4.1) only tests θ_r^* against $\zeta \in K_r$. We now extend it to arbitrary $\zeta \in K$, by moving along the segment from θ_r^* to ζ . Let $\zeta \in K$ and $u \in T(\theta_r^*)$ be arbitrary. For $t \in (0, 1)$, define

$$\zeta_t := t\zeta + (1-t)\theta_r^* \in K$$

(by convexity of K). Since $\zeta_t \rightarrow \theta_r^*$ as $t \rightarrow 0^+$ and $\|\theta_r^*\| < r$ strictly, there is $t_0 \in (0, 1)$ with $\|\zeta_t\| \leq r$ for all $t \in (0, t_0]$. Fix such a t , so $\zeta_t \in K_r$. By (4.1) at this ζ_t ,

$$\langle \beta(\theta_r^*, u), \eta(\zeta_t, \theta_r^*) \rangle + \xi(\zeta_t) - \xi(\theta_r^*) + J^0(\theta_r^*, \zeta_t - \theta_r^*) \geq 0. \quad (4.4)$$

By hypotheses (1) and (2), η is affine in its first argument and $\eta(\theta_r^*, \theta_r^*) = 0$, so

$$\eta(\zeta_t, \theta_r^*) = t\eta(\zeta, \theta_r^*) + (1-t)\eta(\theta_r^*, \theta_r^*) = t\eta(\zeta, \theta_r^*)$$

exactly. By positive homogeneity of the Clarke derivative in its second argument [3],

$$J^0(\theta_r^*, \zeta_t - \theta_r^*) = J^0(\theta_r^*, t(\zeta - \theta_r^*)) = tJ^0(\theta_r^*, \zeta - \theta_r^*)$$

exactly. By convexity of ξ (hypothesis (4)),

$$\xi(\zeta_t) - \xi(\theta_r^*) \leq t(\xi(\zeta) - \xi(\theta_r^*)).$$

Putting these three facts into (4.4),

$$\begin{aligned} 0 &\leq \langle \beta(\theta_r^*, u), \eta(\zeta_t, \theta_r^*) \rangle + \xi(\zeta_t) - \xi(\theta_r^*) + J^0(\theta_r^*, \zeta_t - \theta_r^*) \\ &\leq t[\langle \beta(\theta_r^*, u), \eta(\zeta, \theta_r^*) \rangle + \xi(\zeta) - \xi(\theta_r^*) + J^0(\theta_r^*, \zeta - \theta_r^*)]. \end{aligned}$$

Dividing by $t > 0$,

$$\langle \beta(\theta_r^*, u), \eta(\zeta, \theta_r^*) \rangle + \xi(\zeta) - \xi(\theta_r^*) + J^0(\theta_r^*, \zeta - \theta_r^*) \geq 0.$$

Since $\zeta \in K$ and $u \in T(\theta_r^*)$ were arbitrary, $\theta^* := \theta_r^*$ solves $HVI(\beta\eta\xi)$ on all of K . $\square$

Remark 4.2. Step 3 needed no limiting process and no η -hemicontinuity: the final inequality came from exact algebra, for one fixed t , not from a limit $t \rightarrow 0^+$. This matches the corresponding step in Mahato and Nahak [8, proof of Theorem 3.3]. η -hemicontinuity is still needed elsewhere in the proof, since it is a standing hypothesis of Theorem 2.4, which we apply on each K_r .

Remark 4.3. A related question is whether ϖ -coercivity (Definition 2.6), together with a Kadec–Klee-type property of B , forces every approximating sequence to have a subsequence converging strongly to a solution, once $S \neq \emptyset$. We take up the weak-compactness half of this question in Theorem 5.3 below, and the strong-compactness half in Section 5. The key point of Theorem 4.1 is that its own first step “ S is nonempty” needs only Theorem 2.4 (which needs K bounded) plus Condition 3.1. So it gives existence directly on unbounded K , with no separate compactness argument.

5 Structure of the Solution Set

Theorem 4.1 tells us $S \neq \emptyset$. It says nothing about how large S is, or whether it has any useful topological structure. We answer these questions now, still under Condition 3.1. A similar combination a coercivity condition on an unbounded set, used to get weak compactness of the whole solution set rather than just one solution has recently been used for coupled systems of quasi-variational-hemivariational inequalities [17]. Our Theorem 5.3 below is the analogous statement for the single hemivariational inequality $HVI(\beta\eta\xi)$.

5.1 An A Priori Bound on Every Solution

Condition 3.1 does not just produce one solution inside the ball $\overline{B}(0, R_0)$. It forces every solution to lie there, no matter how that solution was found.

Proposition 5.1 (A priori bound). Under Condition 3.1, every solution $\theta \in S$ satisfies $\|\theta\| < R_0$. Consequently $S \subset K_{R_0} := K \cap \overline{B}(0, R_0)$, so S is a bounded subset of B .

Proof. Let $\theta \in S$. Suppose, for contradiction, that $\|\theta\| \geq R_0$. Since θ solves $HVI(\beta\eta\xi)$, take $\zeta = \theta_0$ in Definition 2.1. For every $u \in T(\theta)$,

$$\langle \beta(\theta, u), \eta(\theta_0, \theta) \rangle + \xi(\theta_0) - \xi(\theta) + J^0(\theta, \theta_0 - \theta) \geq 0.$$

Since $T(\theta) \neq \emptyset$, fix one such u . But $\|\theta\| \geq R_0$, so Condition 3.1 applies at this θ, u and gives the strict reverse inequality

$$\langle \beta(\theta, u), \eta(\theta_0, \theta) \rangle + \xi(\theta_0) - \xi(\theta) + J^0(\theta, \theta_0 - \theta) < 0.$$

This is a contradiction. So $\|\theta\| < R_0$. $\square$

Remark 5.2. Proposition 5.1 does not care how a solution was produced. It applies just as well to the solution built by truncation in Theorem 4.1 as to any other solution that might exist for a different reason. That is what makes it useful for describing all of S , not just one solution.

5.2 Weak Compactness of the Solution Set

Theorem 5.3. Under Condition 3.1 and the hypotheses of Theorems 2.4 and 2.3, the solution set S of $HVI(\beta\eta\xi)$ is nonempty, bounded, weakly closed, and weakly compact.

Proof. Nonemptiness is Theorem 4.1. Boundedness is Proposition 5.1.

For weak closedness: by Theorem 2.3, $\theta \in K$ solves $HVI(\beta\eta\xi)$ if and only if θ satisfies $(*)$ for every $\zeta \in K$. So

$$S = \bigcap_{\zeta \in K} G(\zeta),$$

with $G(\zeta)$ as in Lemma 2.5. Each $G(\zeta)$ is weakly closed, by that lemma. An intersection of weakly closed sets is weakly closed. So S is weakly closed.

Finally, S is a bounded, weakly closed subset of the reflexive space B . By reflexivity, every closed ball $\overline{B}(0, r)$ is weakly compact (Kakutani’s theorem, see, e.g., [7]), and a weakly closed subset of a weakly compact set is weakly compact. So $S \subset \overline{B}(0, R_0)$ is weakly compact. $\square$

Remark 5.4. This proof does not use convexity of S . In fact S need not be convex: convexity of the solution set would need a stronger, symmetric form of monotonicity along the whole segment joining two solutions, beyond $(\eta\xi\sigma\psi - p)$ -monotonicity as defined. We do not pursue conditions for convexity of S here.

5.3 The Strongly Monotone Case: Collapse to a Singleton

When the stronger hypotheses of the uniqueness theorem (Theorem 2.7) hold, the structure question becomes trivial: there is only one solution to describe.

Corollary 5.5. Under the hypotheses of Theorem 5.3, suppose also that conditions $(\dagger)$ and $(\ddagger)$ of Theorem 2.7 hold. Then $S=\{\theta^*\}$ for a unique $\theta^* \in K$ with $\|\theta^*\| < R_0$. In particular, S is compact in the norm topology. Proof. By Theorem 5.3, $S \neq \emptyset$. By Theorem 2.7, $HVI(\beta\eta\xi)$ has at most one solution. So S has exactly one point θ^* , and $\|\theta^*\| < R_0$ by Proposition 5.1. A single point is compact in any Hausdorff topology, including the norm topology. $\square$

5.4 Norm-Compactness in the Non-Unique Case: An Open Problem

Is S compact in the norm topology even when $(\dagger)(\ddagger)$ do not hold, so S may have more than one point? A standard device for upgrading weak convergence of a sequence of solutions to strong convergence is the Kadec-Klee property of B : if $\theta_n \rightharpoonup \theta$ weakly and $\|\theta_n\| \rightarrow \|\theta\|$, then $\theta_n \rightarrow \theta$ in norm (for a recent, unified treatment of this property and its consequences, see [18]). It is tempting to use the same property here. We now explain why norm compactness does not follow from our assumptions without one more ingredient.

Remark 5.6. The Kadec-Klee property turns weak convergence of a given sequence into strong convergence. But it only works once we already know $\|\theta_n\| \rightarrow \|\theta\|$ for that sequence. Take any sequence $\{\theta_n\} \subset S$ with $\theta_n \rightharpoonup \theta \in S$ (such a sequence exists by Theorem 5.3). To use Kadec-Klee, we would need to show $\|\theta_n\| \rightarrow \|\theta\|$ separately. Weak lower semicontinuity of the norm only gives $\|\theta\| \leq \liminf_{n \rightarrow \infty} \|\theta_n\|$, never the reverse inequality. That reverse inequality is exactly what is missing. In classical uses of this device (see, e.g., [4]), the reverse inequality is available when the sequence is a minimizing sequence. A coercivity growth estimate pins down its asymptotic norm in that case. Here, $\{\theta_n\}$ is an arbitrary sequence of exact solutions, not a minimizing sequence. Condition 3.1 is only an inequality at one test point, $\varsigma = \theta_0$. It does not control $\|\theta_n\| - \|\theta\|$ when two solutions θ_n, θ are tested against each other. Getting that kind of control would need a two-point estimate like $(\dagger)$, but valid without the strong-monotonicity constant $c > 0$. That is exactly the case we exclude once we leave the setting of Corollary 5.5.

Open Problem 5.7. Find a hypothesis strictly weaker than $(\dagger)(\ddagger)$ (which force S to be a singleton), but strong enough that, together with a KadecKlee-type property on B , it implies S is compact in the norm topology. Or exhibit an example where S is weakly but not norm compact under Condition 3.1 alone. Equivalently, in the language of the Kuratowski measure of non-compactness μ [2]: find conditions under which $\mu(S)=0$ without forcing S to be a singleton.

6 Illustrative Examples

Example 6.1. Let $B = \mathbb{R}^n$ (finite-dimensional, hence reflexive), and $K = \mathbb{R}_+^n := \{\theta \in \mathbb{R}^n : \theta_i \geq 0\}$, which is nonempty, closed, convex, and unbounded. Let $A \in \mathbb{R}^{n \times n}$ satisfy $\langle A\theta, \theta \rangle \geq c \|\theta\|^2$ for some $c > 0$ and all θ , let $b \in \mathbb{R}^n$, and set (single-valued, so $\beta(\theta, u) = u$) $T(\theta) = A\theta + b$, $\eta(\varsigma, \theta) = \varsigma - \theta$, $\xi \equiv 0$, $J \equiv 0$, $\sigma(\varsigma - \theta) := c \|\varsigma - \theta\|^2$, $\psi \equiv 0$. Then T is $(\eta\xi\sigma\psi - p)$ -monotone with respect to β . It is η -hemicontinuous, since T is continuous and η is linear. Hypotheses (1)(6) of Theorem 4.1 hold trivially: $\xi, J \equiv 0$ are convex and Lipschitz for free; $\sigma, \psi(\cdot, \cdot)$ are continuous, hence lower semicontinuous; and $\theta \mapsto \langle T\theta, \varsigma - \theta \rangle$ is continuous, hence weakly sequentially upper semicontinuous.

For Condition 3.1, take $\theta_0 = 0 \in K$. For $\theta \in K$, $\langle A\theta + b, -\theta \rangle = -\langle A\theta, \theta \rangle - \langle b, \theta \rangle \leq -c \|\theta\|^2 + \|b\| \|\theta\| = \|\theta\| (\|b\| - c \|\theta\|)$, which is strictly negative once $\|\theta\| > \|b\|/c$. So Condition 3.1 holds with $R_0 := \|b\|/c + 1$. Theorem 4.1 then gives a solution $\theta^* \in \mathbb{R}_+^n$ of $\langle A\theta^* + b, \varsigma - \theta^* \rangle \geq 0 \quad \forall \varsigma \in \mathbb{R}_+^n$, the classical linear complementarity/variational inequality on the nonnegative orthant [4].

Now check $(\dagger)$ and $(\ddagger)$. Since $J \equiv 0$ here, $J^0 \equiv 0$, and for $(\dagger)$: $\sigma(\varsigma - \theta) + \psi(\theta, \varsigma) + \sigma(\theta - \varsigma) + \psi(\varsigma, \theta) - J^0(\theta, \varsigma - \theta) - J^0(\varsigma, \theta - \varsigma) = 2c \|\varsigma - \theta\|^2 - 0 - 0 = 2c \|\varsigma - \theta\|^2$, so $(\dagger)$ holds with equality. For $(\ddagger)$: $\langle A\theta_1 + b, \theta_2 - \theta_1 \rangle + \langle A\theta_2 + b, \theta_1 - \theta_2 \rangle = -\langle A(\theta_1 - \theta_2), \theta_1 - \theta_2 \rangle \leq -c \|\theta_1 - \theta_2\|^2 \leq 0$, so $(\ddagger)$ holds too. By Corollary 5.5, $S = \{\theta^*\}$. This matches the classical fact that a coercive monotone affine variational inequality has a unique solution. Here $\mathbb{R}^n$ is finite-dimensional, so weak and norm topologies coincide on bounded sets. This example is too simple to test Open Problem 5.7: a genuine, infinite-dimensional test would need S to have more than one point, and that cannot happen here once $(\dagger)(\ddagger)$ hold.

Example 6.2. Keep $B = \mathbb{R}^n$, $K = \mathbb{R}_+^n$, $T(\theta) = A\theta + b$, $\beta(\theta, u) = u$, $\eta(\varsigma, \theta) = \varsigma - \theta$, and A, b as in Example 6.1, but now set $\sigma \equiv 0$, $\psi \equiv 0$, and $\xi(\theta) := \gamma \|\theta\|_1$, $J(\theta) := \delta \|\theta\|_\infty$, $\gamma, \delta \geq 0$ fixed constants. Both ξ and J are finite everywhere, hence continuous, convex, and Lipschitz. So ξ is convex and lower semicontinuous (hypothesis (4)), and J is locally Lipschitz (hypothesis (5)). Since J is convex, its Clarke derivative agrees with the ordinary directional derivative, $J^0(\theta, d) = \lim_{t \rightarrow 0^+} [J(\theta + td) - J(\theta)]/t$. J fails to be differentiable exactly where the maximizing coordinates of $|\theta_i|$ are not unique for instance at $\theta = 0$. So J is a genuine nonsmooth potential, not just a formal device.

For Condition 3.1, again take $\theta_0 = 0$. Since $J = \delta \|\cdot\|_\infty$ is convex and positively homogeneous, its subdifferential lies in δ times the unit ball of the dual norm $\|\cdot\|_1$, so $J^0(\theta, -\theta) \leq \delta \|\theta\|_\infty \leq \delta \|\theta\|$. Also $\xi(\theta) = \gamma \|\theta\|_1 \geq 0$,

so $\xi(\theta_0) - \xi(\theta) \leq 0$. Combining with the linear-part estimate from Example 6.1, $\langle A\theta + b, -\theta \rangle + \xi(\theta_0) - \xi(\theta) + J^0(\theta, -\theta) \leq \|\theta\|(\|b\| + \delta - c\|\theta\|)$, which is strictly negative once $\|\theta\| > (\|b\| + \delta)/c$. So Condition 3.1 holds with $R_0 := (\|b\| + \delta)/c + 1$, no matter what γ is. Theorem 4.1 gives a solution $\theta^* \in \mathbb{R}_+^n$ of a genuine hemivariational inequality with an ℓ^1 penalty and a nonsmooth ℓ^∞ potential a problem that does not reduce to a classical variational inequality. Condition 3.1 holds, and the hypotheses of Theorem 5.3 hold too, so S is weakly compact here as well.

With $\sigma \equiv 0$, condition $(\dagger)$ fails. Its left side reduces to $-J^0(\zeta, \theta) - J^0(\zeta, \theta - \zeta)$, which is bounded by $2\delta\|\zeta - \theta\|$ in absolute value. So it cannot dominate $2c\|\zeta - \theta\|^2$ once $\|\zeta - \theta\|$ is large. So Corollary 5.5 does not apply, and S might contain more than one point. (Unlike in Example 6.1, $(\dagger)$ now involves J^0 through the mixed term in Theorem 2.7. So simply restoring $\sigma(\zeta - \theta) := c\|\zeta - \theta\|^2$ no longer forces $(\dagger)$ for free once $\delta > 0$: the J^0 -terms grow only linearly while σ grows quadratically, so $(\dagger)$ would hold once $\|\zeta - \theta\|$ is large, but it would need a separate check near $\zeta = \theta$. We simply keep $\sigma \equiv \psi \equiv 0$ here. This keeps the illustration of the non-unique regime clean, whatever δ is.)

Example 6.3. Let $B = \ell^2$ with its usual inner product and norm; then B is a real reflexive Hilbert space. Denote by P_1 and P_2 the orthogonal projections onto the odd and the even coordinates, so that $\theta = P_1\theta + P_2\theta$ and $\|\theta\|^2 = \|P_1\theta\|^2 + \|P_2\theta\|^2$ for every θ . Take the constraint set to be the whole space, $K := \ell^2$, which is nonempty, closed, convex, and unbounded. Fix a constant $\gamma > 0$ and set $\xi(\theta) := (\max\{0, \|P_1\theta\| - 1\})^2 + \gamma\|P_2\theta\|^2$, $J \equiv 0$, $\sigma \equiv 0$, $\psi \equiv 0$, $\eta(\zeta, \theta) := \zeta - \theta$, $\beta(\theta, u) := u$, $T(\theta) := \{0\} \subset B^* = \ell^2$. The scalar function $t \mapsto (\max\{0, t - 1\})^2$ is convex, nonnegative, and nondecreasing on $[0, \infty)$, and $\theta \mapsto \|P_1\theta\|$ is convex. So their composition is convex, and ξ is convex and continuous on K . Now check that T is $(\eta\xi\sigma\psi - p)$ -monotone with respect to β , and η -hemicontinuous. The only selection of T is the constant zero map. With $u = 0 \in T(\theta)$, the premise of the monotonicity implication reduces to $\xi(\zeta) - \xi(\theta) \geq 0$, and its conclusion reduces to the same inequality $\xi(\zeta) - \xi(\theta) \geq \sigma(\zeta - \theta) + \psi(\theta, \zeta) = 0$. So the implication is automatic, and the hemicontinuity is trivial. Hypotheses (1)(6) of Theorem 4.1 hold as well: $\eta(\theta, \theta) = 0$ and η is affine in its first argument; σ and ψ are continuous, hence lower semicontinuous; ξ is convex and lower semicontinuous; J is locally Lipschitz; and for each fixed $\zeta \in K$ and $v \in T(\zeta)$ the map $\theta \mapsto \langle \beta(\theta, v), \eta(\zeta, \theta) \rangle = \langle 0, \zeta - \theta \rangle = 0$ is constant, hence weakly sequentially upper semicontinuous.

For Condition 3.1, take $\theta_0 = 0 \in K$ and $R_0 = 2 > \|\theta_0\|$. Since $T(\theta) = \{0\}$, $J^0 \equiv 0$, and $\xi(0) = 0$, the left-hand side of (1) equals $\langle \beta(\theta, 0), \eta(\theta_0, \theta) \rangle + \xi(\theta_0) - \xi(\theta) + J^0(\theta, \theta_0 - \theta) = -\xi(\theta)$. If $\|\theta\| \geq R_0$ then $\xi(\theta) > 0$. Indeed, $\xi(\theta) = 0$ forces $\|P_1\theta\| \leq 1$ and $P_2\theta = 0$, hence $\|\theta\| = \|P_1\theta\| \leq 1 < 2$, a contradiction. So Condition 3.1 holds with $R_0 = 2$, and Theorem 4.1 yields at least one solution $\theta^* \in \ell^2$.

An element $\theta \in K$ solves $HVI(\beta\eta\xi)$ exactly when $\xi(\zeta) - \xi(\theta) \geq 0$ for every $\zeta \in K$ that is, when θ minimizes ξ over K . Since $\xi \geq 0$, and $\xi(\theta) = 0$ if and only if $P_2\theta = 0$ and $\|P_1\theta\| \leq 1$, the solution set is $S = \{\theta \in \ell^2 : P_2\theta = 0, \|P_1\theta\| \leq 1\}$, the closed unit ball of the infinite-dimensional subspace $P_1(\ell^2)$. Proposition 5.1 places S inside K_{R_0} ; this is immediate here too, since $\|\theta\| \leq 1 < R_0$ on S . S is convex and norm-closed, hence weakly closed. It sits inside the weakly compact ball $\overline{B}(0, 1)$ in the reflexive space ℓ^2 , so it is weakly compact in agreement with Theorem 5.3.

The set S is not compact in the norm topology and contains more than one point. If e_{2k-1} denotes the standard basis vector supported on the $(2k-1)$ -st coordinate, then $e_{2k-1} \in S$ for every k , while $\|e_{2k-1} - e_{2m-1}\| = \sqrt{2}$ ($k \neq m$), so no subsequence of (e_{2k-1}) converges in norm, although $e_{2k-1} \rightarrow 0 \in S$ weakly. Here $\sigma \equiv 0$, so condition $(\dagger)$ of Theorem 2.7 fails, and Corollary 5.5 does not apply. This example realizes the non-unique regime of Open Problem 5.7 in a genuine infinite-dimensional space. It also shows that the weak compactness from Theorem 5.3 cannot, in general, be upgraded to norm compactness without the extra two-point estimate discussed in Remark 5.6.

7 Conclusion

We proved two things about $HVI(\beta\eta\xi)$ on unbounded K . First, a concrete coercivity condition (Condition 3.1) is enough for existence (Theorem 4.1). The proof is a truncation argument, adapted from Mahato and Nahak [8] to the penalty term ξ and the Clarke derivative J^0 . Second, the same condition controls the whole solution set: every solution lies in one fixed ball, and S is weakly compact (Theorem 5.3). Under strong monotonicity, S is a single point, so norm-compactness is trivial (Corollary 5.5). Without strong monotonicity, we do not claim norm-compactness. We explained precisely which two-point estimate is missing (Remark 5.6), and we record the gap as an open problem (Open Problem 5.7).

Some limitations remain. Condition 3.1 is sufficient for existence, but not necessary; a weaker, direction-dependent condition might also work, at the cost of a harder truncation argument. Theorem 4.1 gives a solution, with no information on the full solution set beyond Theorem 5.3 and Corollary 5.5. Combining Condition 3.1 with a genuinely weaker-than- $(\dagger)$ hypothesis, as asked in Open Problem 5.7, is a natural next

step. Finally, we have not developed a numerical method for approximating θ^* on infinite-dimensional, unbounded K . Combining the truncation idea with a projected-subgradient or proximal scheme for ξ and J^0 would be needed before the results could be used computationally.

References

- [1] K. Fan, "A generalization of Tychonoff's fixed point theorem," *Math. Ann.*, vol. 142, pp. 305-310, 1961.
- [2] K. Kuratowski, *Topology*, vols. 12, Academic Press, New York, 1968.
- [3] F. H. Clarke, "Generalized gradients of Lipschitz functionals," *Adv. Math.*, vol. 40, pp. 526-7, 1981.
- [4] E. Zeidler, *Nonlinear Functional Analysis and its Applications II/B*, Springer, New York, 1990.
- [5] B. Lee, M. F. Khan, and Salahuddin, "Hybrid-type set-valued variational-like inequalities in reflexive Banach spaces," *J. Appl. Math. Inform.*, vol. 27, pp. 1371-1389, 2009.
- [6] N. Costea and C. Lupu, "On a class of variational-hemivariational inequalities involving set valued mappings," *Adv. Pure Appl. Math.*, vol. 1, no. 2, pp. 233-246, 2010.
- [7] H. Brézis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Springer, New York, 2011.
- [8] N. K. Mahato and C. Nahak, "Variational-like inequalities and equilibrium problems with generalized monotonicity in Banach spaces," *Adv. Oper. Res.*, vol. 2012, Article ID 648070, 15 pages, 2012.
- [9] A. E. Hashoosh, M. Alimohammady, and M. K. Kalleji, "Existence results for some equilibrium problems involving α -monotone bifunction," *Int. J. Math. Math. Sci.*, vol. 2016, pp. 1-5, 2016.
- [10] A. E. Hashoosh, "Existence and uniqueness solutions for a class of hemivariational inequalities," *J. Math. Inequal.*, vol. 11, pp. 565-576, 2017.
- [11] A. A. Alwan, H. A. Naser, and A. E. Hashoosh, "The solutions of mixed hemiequilibrium problem with application in Sobolev space," *IOP Conf. Ser. Mater. Sci. Eng.*, vol. 571, Art. 012019, 2019. <https://doi.org/10.1088/1757-899X/571/1/012019>.
- [12] A. E. Hashoosh and A. M. Jebur, "Improvement of well-posedness for hemiequilibrium problems," *Math. Eng. Sci. Aerosp.*, vol. 12, pp. 5-17, 2021.
- [13] W. S. Abed, A. E. Hashoosh, and D. M. Abbas, "Existence results for some equilibrium problems involving B_h -monotone trifunction," *AIP Conf. Proc.*, vol. 2414, Art. 040015, 2023.
- [14] A. K. Khisbag, A. E. Hashoosh, and H. G. Kalt, "A new class of invexity function for vector variational problems involving H-spaces," in *AIP Conf. Proc.*, vol. 2414, no. 1, pp. 1-15, 2023.
- [15] A. E. Hashoosh, E. M. Hameed, and S. D. Mohsen, "A new type of monotonous function using KKM-mapping involving hemivariational inequality," *Asia Pac. J. Math.*, vol. 11, Art. 94, pp. 1-9, 2024.
- [16] S. Carl, "Quasilinear parabolic variational-hemivariational inequalities in $\mathbb{R}^N \times (0, \tau)$ under bilateral constraints," *Commun. Anal. Mech.*, vol. 17, no. 1, pp. 4160, 2025.
- [17] J. Cen, N. Costea, D. Dumitrescu, and M. Stanciu, "A set-valued approach for quasi-variational hemivariational inequality systems," *J. Fixed Point Theory Appl.*, vol. 27, no. 3, Article 64, 2025.
- [18] W. M. Kozłowski, "A note on the KadecKlee property," *Axioms*, vol. 15, no. 2, Article 115, 2026.
- [19] H. M. B. Alrikabi, L. J. Barghooth, A. K. Muazi, and A. E. Hashoosh, "A Fan-KKM approach to vector hemivariational-like inequalities with ζ - θ -monotonicity," *Bol. Soc. Parana. Mat.*, vol. 44, pp. 1-9, 2026.
- [20] H. H. Muttair and A. E. Hashoosh, "Existence results for α -nonstandard equilibrium trifunction," *Asia Pac. J. Math.*, vol. 13, Art. 133, pp. 1-10, 2026.