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Some outcomes of fixed Point theorems pertaining to *C*-Class Functions within *G*-Metric Spaces

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Abstract
We demonstrate that there is and only is one self-mapping $f: X \to X$ that satisfies a contractive condition under the management of three control functions M_1, M_2, M_3 and an auxiliary function N using Ansari's C-class function [1] and Khan, Swaleh, and Sessa's altering distance functions [11]. This study presents a G-metric counterpart to Huang, Deng, and Radenović's b-metric results [10]. The proof relies on a contradiction argument to control the auxiliary functions M_i along the Picard sequence, as the rectangle inequality of a G-metric does not carry a multiplicative coefficient, unlike the triangle inequality of a b-metric. We construct several corollaries that recover classical contraction-type theorems, provide an example, as well as how to apply them to the problem of finding a unique solution to a nonlinear integral equation.

Keywords: G-metric space; C-class function; Iteration process; fixed point; nonlinear equation.

1 introduction and historical background

Metric fixed point theory is based on the Banach contraction principle, which says that every contraction defined on a full metric space has a unique fixed point, and the Picard iterates converge to that point. The principle has been generalised in two main ways that are mostly separate from each other: (a) generalising the ambient space and (b) generalising the contractive condition itself. This is because it is simple and can be used in a lot of different types of differential and integral equations. There are results in this paper that fall in the middle of both ways.

In the early 1960s, S. Gähler [6] created the concept of 2-metric spaces, which was the first attempt to enrich the metric structure by enabling a "distance" to depend on more than two points. This concept was the first attempt to enrich the metric structure. It was necessary for a function $d: X \times X \times X \to \mathrm{R}^+$ to vanish precisely when at least two of its arguments coincide, as well as to fulfil symmetry and an inequality of the tetrahedral type. Despite the fact that 2-metric spaces are appealing from a geometric standpoint, it has been discovered that they behave badly with regard to continuity. It is not necessary for $d(x, y, \cdot)$ to be continuous, and a 2-metric is not generally induced by an ordinary metric, nor is it reducible to it.

In response to these challenges, Dhage [5] put forward the concept of a *D-metric space* in his doctoral thesis. Dhage and a number of other authors who came after him established a topological and fixed point theory for D-metric spaces. On the other hand, Mustafa and Sims [12] demonstrated that the majority of the topological assumptions asserted for D-metric spaces are, in fact, wrong. More specifically, the topology that they create does not necessarily have to be Hausdorff, and sequences can have more than one limit. In order to rectify these flaws, Mustafa and Sims [13] presented the concept of a *G-metric space*. This function must satisfy a slightly different and carefully chosen set of axioms, which can be found in Definition 2.1 below. This set of axioms ensures that the induced topology is Hausdorff and metrisable in a controlled sense. We would like to direct the reader to [13] and the references contained within it for the underlying theory. This solved the structural defects of D-metric spaces while preserving their geometric attractiveness. Additionally, it sparked a huge body of fixed point results in the G-metric setting(see [9, 8]).

On the side of the contractive condition, Khan, Swaleh, and Sessa [11] presented the concept of a *altering distance function* in 1984. This function is a continuous, non-decreasing mapping $\psi: \mathrm{R}^+ \to \mathrm{R}^+$, and $\psi(t)$ is equal to zero if and only if t is equal to zero. A number of nonlinear contractive conditions can be unified under a single inequality of the form $\psi\big(d(fx, fy)\big) \le k\, \psi\big(d(x, y)\big)$ during substituting the identity map in the classical contraction inequality $d(fx, fy) \le k\, d(x, y)$ with a control function ψ. This allows one to

"alter" the way distances are compared, and it also allows one to "alter" the way in which distances are compared.
Ansari [1] was able to accomplish a second, more abstract layer of unification. He did this by introducing the concept of a *C-class function*, which is a continuous mapping $F:\mathrm{R}^+ \times \mathrm{R}^+ \to \mathrm{R}$ that satisfies the condition $F(s,t) \le s$ for all $s,t \ge 0$. Additionally, the property that $F(s,t) = s$ ensures that s or t becomes equal to zero. Example 2.5 demonstrates that this single axiomatic framework encompasses a wide range of contraction types that are seen in the literature, including linear contractions, Boyd-Wong type contractions, weak contractions, and many more sorts of contractions. All of these contraction types correspond to particular choices of F. Several studies have investigated variational inequalities and related problems, including hemivariational inequalities, equilibrium problems, and nonlocal problems [15, 16, 17, 18, 19].

1.1 The aim of this article

They combined C-class functions with changing distance functions and came up with a unified fixed-point theorem for *b-metric spaces* (Bakhtin [2], Czerwik [4]). It was based on three control functions (M_1, M_2, M_3) and a fourth function (N). They then used their results to solve ordinary differential and nonlinear integral equations. Because a b-metric meets the proof in [10] has to use this coefficient in every estimate. To allow contraction constants outside the simple range $[0,1/s)$, they have to do a delicate case analysis (their Lemma 2.1). In G-metric spaces, on the other hand, the rectangle inequality says that $G(x,y,z) \le G(x,a,a) + G(a,y,z)$ *without* any multiplicative coefficient. This means that there is no direct way to translate the b-metric argument; the bookkeeping is different, and as we explain in Remark 3.3, a simple term-by-term estimate of the control functions M_i along the Picard sequence is *not sufficient* and must be replaced by an argument by contradiction. Earlier, Tomar, Sharma, Beloul et al. [14] and Hamaizia and Ansari [7] looked at fixed point theorems combining C-class functions with G-metric spaces in similar but different ways. This paper focuses on the family of control functions M_1, M_2, M_3, N used in [10] and applies them to a G-metric setting and gives a full and rigorous proof of the existence and uniqueness theorem (Theorem 3.5), along with corollaries and an example of how to use it to solve a nonlinear integral equation.
Follow these steps to organize the rest of the paper. The second part of the paper gives some basic information and definitions about G-metric spaces, C-class functions, and changing distance functions. Included in Section 3 are the main results: the control functions M_1, M_2, M_3, N, the key auxiliary lemma, and a full proof of the main fixed point theorem. As a result of specializing the C-class function F, Section 4 lists a number of corollaries. Uses the main result on a nonlinear integral equation in Section 5.

2 Preliminaries

This section provides a summary of the main and supporting findings that will be used throughout the study. We provide a concise overview of the concepts of G-metric spaces, as well as G-completeness, G-convergence, C-class functions, and altering distance functions, for the reader's convenience. We also collect a fundamental property of G-metrics that will serve in a manner similar to the triangle inequality in determining the estimates necessary for the proofs of our primary findings.

Definition 2.1. [13] Consider X be a non-empty set, and let $G: X \times X \times X \to \mathrm{R}^+$ be a function that, for every $r,s,t,a \in X$:
(G1) $G(r,s,t) = 0$ if and only if $r = s = t$;
(G2) $G(r,r,s) > 0$ for all $r,s \in X$ with $r \ne s$;
(G3) $G(r,r,s) \le G(r,s,t)$ for all $r,s,t \in X$ with $t \ne s$;
(G4) $G(r,s,t) = G(r,t,s) = G(s,r,t) = \cdots$ (symmetry in all three arguments);
(G5) $G(r,s,t) \le G(r,a,a) + G(a,s,t)$ (rectangle inequality).
The combination (X,G) is referred to as a G-metric space.
Proposition 2.2. [13] *Consider that (X,G) is a G-metric space. Then for all $r,s,t,a \in X$:*
(i) $G(r,s,s) \ge 0$, and $G(r,s,s) = 0$ if and only if $r = s$;
(ii) $G(r,r,r) = 0$;
(iii) $G(r,s,s) \le G(r,a,a) + G(a,s,s)$.
The G-metric analogue of the conventional triangle inequality $d(r,s) \le d(r,a) + d(a,s)$ is Proposition 2.2(iii), which we will employ frequently.

Definition 2.3. [13] Consider that (X,G) is a G-metric space and $\{r_n\}$ a sequence in X.
(1) $\{r_n\}$ is *G-converges* to $r \in X$ if $G(r_n, r_n, x) \to 0$ as $n \to \infty$.
(2) $\{r_n\}$ is a *G-Cauchy sequence* if $G(r_m, r_n, r_n) \to 0$ as $m,n \to \infty$.
(3) (X,G) is *G-complete* if every G-Cauchy sequence in X is G-convergent in X.

Definition 2.4. [1] A function $F:\mathrm{R}^+ \times \mathrm{R}^+ \to \mathrm{R}$ is called a *C-class* if, for all $s,t \ge 0$,

(i) $F(u,v) \leq u$;
(ii) $F(u,v) = u$ implies $u = 0$ or $v = 0$;
(iii) F is continuous in both variables.
The set $\mathcal{C}$ represents all functions of the C class.
Example 2.5. If the next functions $F:\mathrm{R}^+ \times \mathrm{R}^+ \to \mathrm{R}$ belongs to $\mathcal{C}$ (see [1, 10]):
(1) $F(u,v) = u - v$;
(2) $F(u,v) = \frac{\ln(v+e^u)}{1+v}$;
(3) $F(u,v) = \mu u$, where $0 < \mu < 1$;
(4) $F(u,v) = \frac{u}{(1+v)^\alpha}$, where $\alpha > 0$;
(5) $F(u,v) = u\,\varphi(v)$, where $\varphi:\mathrm{R}^+ \to [0,1)$ is continuous and $\varphi(v) = 1 \Leftrightarrow v = 0$ is replaced by $\varphi(v) < 1$ for $v > 0$;
(6) $F(u,v) = u - \varphi(v)$, where $\varphi:\mathrm{R}^+ \to \mathrm{R}^+$ is continuous and $\varphi(v) = 0 \Leftrightarrow v = 0$;
(7) $F(u,v) = \theta(u)$, where $\theta:\mathrm{R}^+ \to \mathrm{R}^+$ is continuous and $\theta(u) < u$ for all $u > 0$.
Definition 2.6. [11] $\psi:\mathrm{R}^+ \to \mathrm{R}^+$ is a *altering distance function* if
(i) ψ is non-decreasing and continuous;
(ii) $\psi(z) = 0$ if and only if $z = 0$.

3 Principal Findings

We are now going to present the primary findings of this paper. We commence by introducing the auxiliary control functions that serve as the foundation of our contractive condition. These control functions are subsequently estimated using a critical technical lemma that is established during the Picard iteration. This lemma is the indispensable instrument for surmounting the challenge that arises from the unique geometric structure of G-metric spaces, in which the argument employed in the appropriate b-metric setting cannot be directly applied. Lastly, we establish the existence and uniqueness of a fixed point under the proposed contractive condition and derive a number of practical implications.
In the context of a G-metric space, let $f: X \to X$ be a self-mapping expression. As stated in the reference [10], it is possible to declare, for $\alpha, \beta \in X$,M_1

$$(\alpha,\beta) = \max\left\{G(\alpha,\beta,\beta),\ \frac{G(\alpha,f\alpha,f\alpha)\,G(\beta,f\beta,f\beta)}{1+G(\alpha,\beta,\beta)},\ \frac{G(\alpha,f\alpha,f\alpha)\,G(\beta,f\beta,f\beta)}{1+G(f\alpha,f\beta,f\beta)}\right\}, \quad (1)$$

$$M_2(\alpha,\beta) = \max\{G(\alpha,\beta,\beta),\ \frac{G(\alpha,f\alpha,f\alpha)G(\alpha,f\beta,f\beta)+G(\beta,f\beta,f\beta)G(\beta,f\alpha,f\alpha)}{1+G(\alpha,f\alpha,f\alpha)+G(\beta,f\beta,f\beta)},$$
$$\frac{G(\alpha,f\alpha,f\alpha)G(\alpha,f\beta,f\beta)+G(\beta,f\beta,f\beta)G(\beta,f\alpha,f\alpha)}{1+G(\alpha,f\beta,f\beta)+G(\beta,f\alpha,f\alpha)}\}, \quad (2)$$

$$M_3(\alpha,\beta) = \max\{G(\alpha,\beta,\beta),\ \frac{G(\alpha,f\alpha,f\alpha)\,G(\beta,f\beta,f\beta)}{1+G(\alpha,\beta,\beta)+G(\alpha,f\beta,f\beta)+G(\beta,f\alpha,f\alpha)},$$
$$\frac{G(\alpha,f\beta,f\beta)\,G(\alpha,\beta,\beta)}{1+G(\alpha,f\alpha,f\alpha)+G(\beta,f\alpha,f\alpha)+G(\beta,f\beta,f\beta)}\}, \quad (3)$$

and

$$N(\alpha,\beta) = \min\{G(\alpha,f\alpha,f\alpha),\ G(\beta,f\beta,f\beta),\ G(\alpha,f\beta,f\beta),\ G(\beta,f\alpha,f\alpha)\}. \quad (4)$$

Remark 3.1. The phrase $G(\alpha,\beta,\beta)$ represents one of the three terms that are part of each maximum. These terms are contained within each maximum. (1)–(3); then

$$G(\alpha,\beta,\beta) \leq M_i(\alpha,\beta), \qquad i = 1,2,3, \qquad \text{for all } \alpha,\beta \in X.$$

Given that $\alpha_0 \in X$ be arbitrary, and let $\{\alpha_n\}$ be the Picard sequence defined by $\alpha_{n+1} = f(\alpha_n)$, $n \geq 0$. Throughout the entirety of this paragraph, we consider that

$$\alpha_n \neq \alpha_{n+1} \qquad \text{for every } n \geq 0, \qquad (5)$$

if this is not the case, then α_n is a fixed point of f, and there is nothing to show about it. Then,

$$G_n := G(\alpha_n,\alpha_{n+1},\alpha_{n+1}), \qquad n \geq 0.$$

By (5) and (G2), $G_n > 0$ for every $n \geq 0$. Since $f(\alpha_{n-1}) = \alpha_n$ and $f(\alpha_n) = \alpha_{n+1}$, we have

$$G(\alpha_{n-1},f\alpha_{n-1},f\alpha_{n-1}) = G_{n-1}, \qquad G(\alpha_n,f\alpha_n,f\alpha_n) = G_n.$$

Lemma 3.2. *For every $n \geq 1$ and for each $i = 1,2,3$,*

$$M_i(\alpha_{n-1},\alpha_n) \leq \max\{G_{n-1},G_n\}. \qquad (6)$$

Moreover, $M_2(\alpha_{n-1},\alpha_n) \leq G_{n-1}$.
Proof. Throughout the proof we use repeatedly that for $a,b \geq 0$,

$$\frac{ab}{1+a} \leq b \qquad \text{and} \qquad \frac{ab}{1+b} \leq a, \qquad (7)$$

both being immediate from $a/(1+a) < 1$ and $b/(1+b) < 1$.
Case $i = 1$. Substituting $f\alpha_{n-1} = \alpha_n$ and $f\alpha_n = \alpha_{n+1}$ into (1),

$$M_1(\alpha_{n-1},\alpha_n) = \max\left\{G_{n-1},\ \frac{G_{n-1}G_n}{1+G_{n-1}},\ \frac{G_{n-1}G_n}{1+G_n}\right\}.$$

By (7) (with $a = G_{n-1}$, $b = G_n$), $\frac{G_{n-1}G_n}{1+G_{n-1}} \le G_n$, and (with $a = G_n$, $b = G_{n-1}$) $\frac{G_{n-1}G_n}{1+G_n} \le G_{n-1}$. Hence

$$M_1(\alpha_{n-1},\alpha_n) \le \max\{G_{n-1}, G_n, G_{n-1}\} = \max\{G_{n-1}, G_n\}.$$

Case $i = 2$. Using $G(\alpha_{n-1}, f\alpha_{n-1}, f\alpha_{n-1}) = G_{n-1}$, $G(\alpha_n, f\alpha_n, f\alpha_n) = G_n$, $G(\alpha_{n-1}, f\alpha_n, f\alpha_n) = G(\alpha_{n-1}, \alpha_{n+1}, \alpha_{n+1})$ and $G(\alpha_n, f\alpha_{n-1}, f\alpha_{n-1}) = G(\alpha_n, \alpha_n, \alpha_n) = 0$ (by Proposition 2.2(ii)), (2) becomes

$$M_2(\alpha_{n-1},\alpha_n) = \max\{G_{n-1},\ A_n,\ B_n\},$$

where

$$A_n = \frac{G_{n-1}\,G(\alpha_{n-1},\alpha_{n+1},\alpha_{n+1})}{1+G_{n-1}+G_n}, \qquad B_n = \frac{G_{n-1}\,G(\alpha_{n-1},\alpha_{n+1},\alpha_{n+1})}{1+G(\alpha_{n-1},\alpha_{n+1},\alpha_{n+1})}.$$

By Proposition 2.2(iii) (rectangle inequality with $a = \alpha_n$),

$$G(\alpha_{n-1},\alpha_{n+1},\alpha_{n+1}) \le G_{n-1} + G_n.$$

Hence

$$A_n \le \frac{G_{n-1}(G_{n-1}+G_n)}{1+G_{n-1}+G_n} < G_{n-1},$$

since $(G_{n-1}+G_n)/(1+G_{n-1}+G_n) < 1$; and by (7), $B_n \le G_{n-1}$. Therefore

$$M_2(\alpha_{n-1},\alpha_n) = \max\{G_{n-1}, A_n, B_n\} = G_{n-1} \le \max\{G_{n-1}, G_n\}.$$

Case $i = 3$. Using the same substitutions as above, (3) becomes

$$M_3(\alpha_{n-1},\alpha_n) = \max\{G_{n-1},\ T_2,\ T_3\},$$

where

$$T_2 = \frac{G_{n-1}G_n}{1+G_{n-1}+G(\alpha_{n-1},\alpha_{n+1},\alpha_{n+1})}, \qquad T_3 = \frac{G_{n-1}\,G(\alpha_{n-1},\alpha_{n+1},\alpha_{n+1})}{1+G_{n-1}+G_n}.$$

Dropping the non-negative term $G(\alpha_{n-1},\alpha_{n+1},\alpha_{n+1})$ from the denominator of T_2 and applying (7),

$$T_2 \le \frac{G_{n-1}G_n}{1+G_{n-1}} \le G_n.$$

Using $G(\alpha_{n-1},\alpha_{n+1},\alpha_{n+1}) \le G_{n-1}+G_n$ as in Case $i = 2$,

$$T_3 \le \frac{G_{n-1}(G_{n-1}+G_n)}{1+G_{n-1}+G_n} < G_{n-1}.$$

Hence

$$M_3(\alpha_{n-1},\alpha_n) \le \max\{G_{n-1}, G_n, G_{n-1}\} = \max\{G_{n-1}, G_n\}.$$

This proves (6) for $i = 1,2,3$, and the sharper bound for $i = 2$. □

Remark 3.3. It is tempting to try to show directly that $M_i(\alpha_{n-1},\alpha_n) = G_{n-1}$, as one might hope by analogy with the elementary estimates in (7). This is exactly what the bound $T_2 \le G_{n-1}$ would require, but T_2 in fact satisfies only the weaker bound $T_2 \le G_n$ obtained above; asserting $T_2 \le G_{n-1}$ outright would amount to assuming $G_n \le G_{n-1}$ *before it has been established*, i.e. a circular argument. The same remark applies to the first term of M_1. This is why Lemma 3.2 only yields the weaker bound $\max\{G_{n-1}, G_n\}$ for $i = 1,3$, and why the proof of Theorem 3.5 below resolves the ambiguity between G_{n-1} and G_n by an argument by contradiction, exactly as is done for the b-metric coefficient $s = 1$ case in [10, Theorem 2.3].

Remark 3.4. A direct computation, identical in spirit to the one above with the roles of fx, fy at fixed points, shows that if $u, v \in X$ satisfy $fu = u$, $fv = v$, then

$$M_i(u,v) = G(u,v,v), \qquad i = 1,2,3, \qquad N(u,v) = 0,$$

because every term in (1)–(4) other than $G(u,v,v)$ contains a factor $G(u,fu,fu) = G(u,u,u) = 0$ or $G(v,fv,fv) = G(v,v,v) = 0$. This fact will be used in the proof of uniqueness.

Theorem 3.5. *Let (X,G) be a complete G-metric space and let $f: X \to X$ be a continuous self-mapping. Suppose there exist a constant $\lambda > 1$, a constant $\xi \ge 0$, two altering distance functions $\psi, \varphi: \mathrm{R}^+ \to \mathrm{R}^+$ with ψ strictly increasing, and a C-class function $F \in \mathcal{C}$ such that, for some fixed $i \in \{1,2,3\}$,*

$$\psi\big(\lambda\, G(f\alpha, f\beta, f\beta)\big) \le F\Big(\psi\big(M_i(\alpha,\beta)\big), \varphi\big(M_i(\alpha,\beta)\big)\Big) + \xi\, N(\alpha,\beta) \qquad (8)$$

for all $\alpha, \beta \in X$, where M_i and N are defined by (1)–(4). Then f has a unique fixed point $u \in X$, and for every $\alpha_0 \in X$ the Picard sequence $\alpha_{n+1} = f(\alpha_n)$ converges to u with respect to G.

Proof. Step 1: Construction of the Picard sequence. Fix $\alpha_0 \in X$ and define $\alpha_{n+1} = f(\alpha_n)$, $n \ge 0$. If $\alpha_{n_0} = \alpha_{n_0+1}$ for some n_0, then α_{n_0} is a fixed point of f and there is nothing further to prove. We therefore assume (5) holds, and retain the notation G_n of the previous subsection.

Step 2: The estimate $G_n \le \lambda^{-1}G_{n-1}$. Applying (8) with $\alpha = \alpha_{n-1}$, $\beta = \alpha_n$, and using $f\alpha_{n-1} = \alpha_n$, $f\alpha_n = \alpha_{n+1}$,

$$\psi(\lambda G_n) \le F\Big(\psi\big(M_i(\alpha_{n-1},\alpha_n)\big), \varphi\big(M_i(\alpha_{n-1},\alpha_n)\big)\Big) + \xi\, N(\alpha_{n-1},\alpha_n). \qquad (9)$$

From (4), since $G(\alpha_n, f\alpha_{n-1}, f\alpha_{n-1}) = G(\alpha_n,\alpha_n,\alpha_n) = 0$,

$$N(\alpha_{n-1},\alpha_n) = \min\{G_{n-1},\ G_n,\ G(\alpha_{n-1},\alpha_{n+1},\alpha_{n+1}),\ 0\} = 0.$$

Hence (9) reduces to

$$\psi(\lambda G_n) \leq F\left(\psi\big(M_i(\alpha_{n-1},\alpha_n)\big),\varphi\big(M_i(\alpha_{n-1},\alpha_n)\big)\right).$$

Since $F \in \mathcal{C}$ satisfies $F(s,t) \leq s$, and since ψ is non-decreasing, Lemma 3.2 gives

$$\psi(\lambda G_n) \leq \psi\big(M_i(\alpha_{n-1},\alpha_n)\big) \leq \psi(\max\{G_{n-1},G_n\}). \qquad (10)$$

Because ψ is *strictly* increasing, (10) yields

$$\lambda G_n \leq \max\{G_{n-1},G_n\}. \qquad (11)$$

We claim that $\max\{G_{n-1},G_n\} = G_{n-1}$. Indeed, if instead $\max\{G_{n-1},G_n\} = G_n$, then (11) gives $\lambda G_n \leq G_n$, i.e. $(\lambda-1)G_n \leq 0$; since $\lambda > 1$, this forces $G_n \leq 0$, hence $G_n = 0$ and $x_n = x_{n+1}$, contradicting (5). This proves the claim, and (11) becomes

$$\lambda G_n \leq G_{n-1}, \qquad \text{i.e.} \qquad G_n \leq \frac{1}{\lambda} G_{n-1}, \qquad n \geq 1. \qquad (12)$$

Step 3: The Picard sequence is G-Cauchy. Iterating (12),

$$G_n \leq \lambda^{-n} G_0, \qquad n \geq 0,$$

and since $\lambda > 1$, $G_n \to 0$ as $n \to \infty$. For $m > n$, after applying the rectangle inequality (G5) many times, the result is because

$$G(\alpha_n,\alpha_m,\alpha_m) \leq \sum_{k=n}^{m-1} G\left(\alpha_k,\alpha_{k+1},\alpha_{k+1}\right) = \sum_{k=n}^{m-1} G_k \leq G_0 \sum_{k=n}^{m-1} \lambda^{-k} \leq \frac{G_0\,\lambda^{-n}}{1-\lambda^{-1}}.$$

Since $\lambda > 1$, the right-hand side tends to 0 as $n \to \infty$, uniformly in $m > n$; hence $\{\alpha_n\}$ is a G-Cauchy sequence.

Step 4: Existence of a fixed point. Since (X,G) is complete, there is $u \in X$ with $x_n \to u$. By continuity of f, $f(\alpha_n) \to f(u)$; on the other hand $f(\alpha_n) = \alpha_{n+1} \to u$. By uniqueness of G-limits, $f(u) = u$, i.e. u is a fixed point of f.

Step 5: Uniqueness. Let $u, v \in X$ be fixed points of f. Then $N(u,v) = 0$ and $M_i(u,v) = G(u,v,v)$ for $i = 1,2,3$, as noted above. By (8) with $\alpha = u, \beta = v$,

$$\psi\big(\lambda\, G(u,v,v)\big) \leq F\left(\psi\big(G(u,v,v)\big),\varphi\big(G(u,v,v)\big)\right) \leq \psi\big(G(u,v,v)\big).$$

Since ψ is strictly increasing, $\lambda\, G(u,v,v) \leq G(u,v,v)$, and since $\lambda > 1$ this forces $G(u,v,v) = 0$, i.e. $u = v$. Thus the fixed point of f is unique, and Step 4 shows every Picard sequence converges to it. □

4 Corollaries

Specializing the C-class function F in Theorem 3.5 to the examples of Example 2.5 recovers a number of classical contraction-type theorems in G-metric spaces.

Corollary 4.1. *Considering (X,G) is a complete G-metric space and $f: X \to X$ continuous. If there is $\lambda > 1$, $\xi \geq 0$, altering distance functions ψ, φ with ψ strictly increasing, such that*

$$\psi\big(\lambda\, G(f\alpha,f\beta,f\beta)\big) \leq \psi\big(M_i(\alpha,\beta)\big) - \varphi\big(M_i(\alpha,\beta)\big) + \xi N(\alpha,\beta)$$

for all $\alpha,\beta \in X$ and some fixed $i \in \{1,2,3\}$. Then, the function f has a unique fixed point, and every Picard sequence converges to it.

Proof. Take $F(s,t) = s - t$ in Theorem 3.5. □

Corollary 4.2. *Considering (X,G) is a complete G-metric space and $f: X \to X$ continuous. Suppose there exist $\lambda > 1$, $\xi \geq 0$, and a continuous function $\theta: \mathrm{R}^+ \to \mathrm{R}^+$ with $\theta(t) < t$ for all $t > 0$, such that*

$$\lambda\, G(f\alpha,f\beta,f\beta) \leq \theta\big(M_i(\alpha,\beta)\big) + \xi N(\alpha,\beta)$$

for all $\alpha,\beta \in X$ and some fixed $i \in \{1,2,3\}$. Then f has a unique fixed point, and every Picard sequence converges to it.

Proof. Take $\psi(t) = t$ and $F(s,t) = \theta(s)$ in Theorem 3.5; note that $\theta(t) < t$ for $t > 0$ together with continuity guarantees $F \in \mathcal{C}$. □

Corollary 4.3. *Considering (X,G) is a complete G-metric space and $f: X \to X$ continuous. if there is $\lambda > 1$ and $0 < \mu < 1$ in which*

$$\lambda\, G(f\alpha,f\beta,f\beta) \leq \mu\, M_i(\alpha,\beta)$$

for all $\alpha,\beta \in X$ and some fixed $i \in \{1,2,3\}$. Then f has a unique fixed point, and every Picard sequence converges to it.

Proof. Taking $\theta(s) = \mu s$ and $\xi = 0$ in Corollary 4.2. □

Corollary 4.4. *Considering (X,G) is a complete G-metric space and $f: X \to X$ continuous. if there is $\lambda > 1$ in which*

$$\lambda\, G(f\alpha,f\beta,f\beta) \leq \frac{M_i(\alpha,\beta)}{1 + M_i(\alpha,\beta)}$$

for all $\alpha, \beta \in X$ and some fixed $i \in \{1,2,3\}$. Then f has a unique fixed point, and every Picard sequence converges to it.
Proof. Taking $\theta(s) = s/(1+s)$ and $\xi = 0$ in Corollary 4.2. □

Corollary 4.5. *Considering (X, G) is a complete G-metric space and $f: X \to X$ continuous. if there is $\lambda > 1$ in which*

$$\lambda\, G(f\alpha, f\beta, f\beta) \le \ln\big(1 + M_i(\alpha, \beta)\big)$$

for all $\alpha, \beta \in X$ and some fixed $i \in \{1,2,3\}$. Then f has a unique fixed point, and every Picard sequence converges to it.
Proof. Take $\theta(s) = \ln(1+s)$ and $\xi = 0$ in Corollary 4.2. □

Example 4.6. Let $X = [0,1]$ with the usual metric $d(\alpha, \beta) = |\alpha - \beta|$, and let G be the G-metric induced by d,

$$G(\alpha, \beta, z) = d(\alpha, \beta) + d(\beta, z) + d(z, \alpha), \qquad \alpha, \beta, z \in X,$$

which is a standard example of a G-metric [13]. Note that $G(\alpha, \beta, \beta) = 2\, d(\alpha, \beta) = 2|\alpha - \beta|$, and that (X, G) is G-complete since (X, d) is complete. Define $f: X \to X$ by $f(\alpha) = \alpha/2$. For $\alpha, \beta \in X$,

$$G(f\alpha, f\beta, f\beta) = 2|f\alpha - f\beta| = |\alpha - \beta| = 1/2\ G(\alpha, \beta, \beta) \le 1/2\ M_i(\alpha, \beta)$$

for $i = 1,2,3$, by Remark 3.1. Hence, for *any* $\lambda \in (1,2)$, setting $\mu = \lambda/2 \in (1/2,1)$ gives

$$\lambda\, G(f\alpha, f\beta, f\beta) \le \frac{\lambda}{2} M_i(\alpha, \beta) = \mu\, M_i(\alpha, \beta),$$

so Corollary 4.3 applies (e.g. with $\lambda = 3/2$, $\mu = 3/4$). Hence f has a unique fixed point, namely $u = 0$, and indeed $f(0) = 0$.

5 Application to a Nonlinear Integral Equation

In the spirit of the program that was presented in [10] for b-metric spaces, we discuss the existence of a solution of the nonlinear integral problem and the uniqueness of that answer. This is an application of Theorem 3.5.

$$\alpha(t) = \int_p^q k\left(t, s, \alpha(s)\right) ds + h(t), \qquad t \in [p, q], \qquad (13)$$

where $-\infty < p < q < \infty$, $h \in C([p,q],\mathrm{R})$, and $k: [p,q] \times [p,q] \times \mathrm{R} \to \mathrm{R}$ is a mapping function satisfying a Lipschitz-type condition in its third argument: there exists $L > 0$ such that

$$|k(t,s,u) - k(t,s,v)| \le L\,|u - v| \qquad \text{for all } t, s \in [p,q],\ u, v \in \mathrm{R}. \qquad (14)$$

Theorem 5.1. *Suppose (14) holds with $L(q-p) < 1$. In the case of the integral equation (13), it is possible to find a singular solution in the set $C([p,q],\mathrm{R})$.*
Proof. Considering $X = C([p,q],\mathrm{R})$ with the supremum norm $\|\cdot\|_\infty$, and define

$$G(\alpha, \beta, z) = \| \alpha - \beta \|_\infty + \| \beta - z \|_\infty + \| z - \alpha \|_\infty, \qquad \alpha, \beta, z \in X.$$

As in Example 4.6, G is the G-metric induced by the metric $\|\cdot - \cdot\|_\infty$ on X, so (X, G) is a G-metric space; it is G-complete because $(X, \|\cdot\|_\infty)$ is a Banach space. Note again that $G(\alpha, \beta, \beta) = 2\, \| \alpha - \beta \|_\infty$.
Define $f: X \to X$ by

$$(f\alpha)(t) = \int_p^q k\left(t, s, \alpha(s)\right) ds + h(t), \qquad t \in [p, q].$$

Since k and h are continuous, $f\alpha \in C([p,q],\mathrm{R})$ for every $\alpha \in X$, so f is well defined; moreover, f is continuous (indeed Lipschitz, see below) with respect to $\|\cdot\|_\infty$, hence continuous with respect to G.
For $\alpha, \beta \in X$ and $t \in [p,q]$, using (14),

$$|(f\alpha)(t) - (f\beta)(t)| \le \int_p^q \big|k\big(t,s,\alpha(s)\big) - k\big(t,s,\beta(s)\big)\big|\ ds \le L \int_p^q |\alpha(s) - \beta(s)|\ ds \le L(q-p)\ \| \alpha - \beta \|_\infty.$$

Taking the supremum over $t \in [p,q]$,

$$\| f\alpha - f\beta \|_\infty \le \mu\ \| \alpha - \beta \|_\infty, \qquad \mu := L(q-p) < 1. \qquad (15)$$

Consequently,

$$G(f\alpha, f\beta, f\beta) = 2\ \| f\alpha - f\beta \|_\infty \le 2\mu\ \| \alpha - \beta \|_\infty = \mu\, G(\alpha, \beta, \beta) \le \mu\, M_i(\alpha, \beta)$$

for $i = 1,2,3$, by Remark 3.1.
Set $\lambda = 1/\sqrt{\mu} > 1$ (recall $0 < \mu < 1$), $\psi(t) = t$, $\varphi(t) = t$, $\xi = 0$, and $F(s,t) = \sqrt{\mu}\, s$; since $0 < \sqrt{\mu} < 1$, $F \in \mathcal{C}$ by Example 2.5(3). Then

$$\begin{aligned}\psi\big(\lambda\, G(f\alpha, f\beta, f\beta)\big) &= \lambda\, G(f\alpha, f\beta, f\beta) \le \lambda\mu\, M_i(\alpha, \beta) = \sqrt{\mu}\, M_i(\alpha, \beta) \\ &= F\left(\psi\big(M_i(\alpha, \beta)\big), \varphi\big(M_i(\alpha, \beta)\big)\right) + \xi N(\alpha, \beta),\end{aligned}$$

so the contractive condition (8) of Theorem 3.5 holds. Hence f has a unique fixed point $u \in C([p,q],\mathrm{R})$, which is precisely a solution of (13), and this solution is unique in $C([p,q],\mathrm{R})$. □

Remark 5.2. As in [10], the Picard sequence $\alpha_{n+1} = f(\alpha_n)$ converges uniformly on $[p,q]$ to the solution u of (13), for any choice of starting function $\alpha_0 \in C([p,q],\mathrm{R})$; this provides, in addition to the abstract existence and uniqueness statement, a constructive method of approximation for the solution.

6 Conclusion

6.1 Conclusion

We have translated the b-metric fixed point theorem of Huang, Deng, and Radenovicć [10], which was originally formulated in terms of C-class functions and modified by modifying distance functions. The control functions M_1, M_2, M_3 along the Picard sequence cannot, in general, be shown to equal G_{n-1} by a direct term-by-term estimate because G-metric spaces do not carry a multiplicative coefficient analogous to the b-metric coefficient s (Remark 3.3). Rather, the bound $\max\{G_{n-1}, G_n\}$ is obtained, and the ambiguity is resolved by an argument by contradiction using $\lambda > 1$. The existence and uniqueness theorem (Theorem 3.5) maintains its full generality with this correction in place. Additionally, the result applies to guarantee the existence and uniqueness of solutions to nonlinear integral equations of Volterra type, and several well-known contraction principles in G-metric spaces are recovered as corollaries.

6.2 Recommendations

In the future, research might concentrate on expanding the current framework to encompass metric-type spaces that are more broad. These spaces could include G_b −metric, partial G-metric, and fuzzy G −metric spaces. Additionally, it would be beneficial to explore multivalued and cyclic mappings from the perspective of the C −class framework that has been proposed. Furthermore, the fixed point theorem that was discovered can be utilised for the solution of a wider range of nonlinear problems, including Fredholm and fractional integral equations, differential equations, and other dynamical systems that are directly related to these. In conclusion, one way to further widen the application of the present findings is to investigate weaker contractive assumptions and new families of C −class functions.

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