SvedbergOpen
DISSEMINATION OF KNOWLEDGE

International Journal of Artificial Intelligence and Machine Learning

Publisher's Home Page: https://www.svedbergopen.com/

Research Paper Open Access

Iterative Approximation For σ-nonstandard Equilibrium Trifunction

Haneen Hameed Muttair [1], Ayed. E. Hashoosh [2]

[1] College of Computer Science and Information Technology University of Sumer Thi Qar, Iraq. E-mail: haneen.hameed@uos.edu.iq
[2]Department of Mathematics, University of Thi-Qar, Iraq. E-mail: Ayed.Hashoosh@utq.edu.iq

Abstract
We describe a new class of equilibrium issues, the σ-nonstandard equilibrium trifunction variational inequality (NSET). It employs a multivalued operator T in the trifunction formulation. To overcome this problem, we can design a three-step iterative algorithm and apply the auxiliary principle technique (AUXPT). The method does not require projection. We construct a thorough error estimate and demonstrate that the iterative sequence converges to a solution when F has three points of monotonicity. This framework contains specific examples of several well-known types of variational inequality and the equilibrium issues, such as the generalized mixed quasi trifunction variational inequality and the generalized equilibrium problem.

Keywords: Trifunction Equilibrium problem; non standard Equilibrium problem variational inequality; Auxiliary principle technique; Convergence.

1 Introduction

This paper studies σ-nonstandard equilibrium trifunction variational inequalities. This framework brings together several key ideas from applied mathematics: variational theory, variational inequalities, and equilibrium problems. It gives a single tool for problems that were once studied separately, each with its own long and complex method. The framework covers several special cases, including obstacle problems, economic models, and problems in mechanics. This unified approach lets researchers find one general solution instead of solving each problem on its own. This research also applies the (AUXPT). This technique is an effective way to build iterative schemes for problems that are hard to solve directly. It turns a theoretical problem into a computable algorithm, giving precise numerical results. The paper also studies monotonicity conditions that improve accuracy and convergence, and give clearer results than earlier work. This does more than advance the pure theory: it also shows that the solutions are stable, and it avoids some common practical difficulties.

In 1964, Stampacchia introduced the theory of variational inequalities [1]. This work gave a new framework for analyzing problems in elasticity and optimization. In 1970, Martinet developed regularization methods based on successive approximation [2]. In 1976, Mosco extended this theory to quasi-variational inequalities [3], where the convex set itself can depend on the solution. In 1981, Glowinski and his colleagues introduced the auxiliary principle technique [4]. This technique can solve hemivariational inequalities, which classical projection methods cannot handle. In 1988, Noor extended nonlinear variational inequalities to elastostatics [5]. In 1994, Blum and Oettli connected optimization with equilibrium problems [16]. In 2003 and 2004, Noor developed the auxiliary principle technique further for equilibrium problems [6] [7]. In 2009, research on nonconvex variational inequalities and extragradient algorithms grew [8]. In 2010, trifunction variational inequalities were introduced, along with iterative methods for solving them [9]. In 2025, Gupta et al. introduced a class of trifunction variational inequalities for general mixed quasi-equilibrium problems [10]. Their formulation adds a nonlinear term $\varphi(\cdot,\cdot)$. They also gave an iterative scheme with three constants ρ, β, γ, and showed that this new class contains several earlier classes as special cases.

Related to this equilibrium framework, several other works have studied monotone trifunctions and bifunctions, nonlocal problems, and hemiequilibrium and hemivariational inequalities. Abed, Hashoosh, and Abbas established existence results for equilibrium problems involving a B_h-monotone trifunction [18]. Alrikabi, Hashoosh, and Alkhalidi obtained existence results for nonlocal problems of (p,q)-Kirchhoff form [19]. Alwan, Naser, and Hashoosh solved the mixed hemiequilibrium problem, with an application in a Sobolev space setting [20]. Hashoosh proved existence and uniqueness of solutions for a class of hemivariational inequalities [21], and, together with Alimohammady and Kalleji, studied equilibrium problems involving an α-monotone bifunction [17]. More recently, Hashoosh and Ali proposed a modern fixed-point method in $\mathcal{G}$-spaces with an application [22], and Hashoosh and Jebur improved the well-posedness theory for hemiequilibrium problems [23].

This work presents a novel class of problems: σ-nonstandard equilibrium trifunction variational inequalities. This class uses a multivalued operator T. We build a new iterative algorithm (Algorithm 3.1) based on the auxiliary principle technique. This algorithm solves the problem without projection. Our framework is unified and flexible: it contains many known classes as special cases, and it applies to problems in engineering, economics, and physics. We also prove convergence under a monotonicity condition that is weaker than those used in earlier work.

2 PRELIMINARIES

Throughout this paper, let H denote a real Hilbert space endowed with the norm $\|.\|$ and the inner product$\langle .,. \rangle$. Let $C(H)$ represent the family of all nonempty compact subsets of H, and let $T: H \to C(H)$ be a set-valued mapping. Assume that $\mathcal{K}$ a nonempty, closed, and convex subset of H, and let $\sigma: \mathcal{K} \times \mathcal{K} \to R$ be a continuous bifunction. For a given trifunction $\mathfrak{F}: \mathcal{K} \times \mathcal{K} \times \mathcal{K} \to H$, our objective is to find a point $\mu \in \mathcal{K}$ such that

$$\sup_{\propto \in T\mu} \mathfrak{F}(\mu, \propto, \hbar) + \sigma(\mu, \hbar) \geq 0 \quad \forall \hbar \in \mathcal{K} \tag{1}$$

which is called the σ-nonstandard equilibrium trifunction in short (NSET) .

1-If $\sup_{\propto \in T\mu} \mathfrak{F}(\mu, \propto, \hbar) \equiv \mathfrak{F}(\hbar, \hbar - \mu)$, and $\sigma(\mu, \hbar) \equiv \phi(\hbar, \mu) - \phi(\mu, \mu)$, then problem (1) is equivalent to finding $\mu \in \mathcal{K}$ such that

$$\mathfrak{F}(\hbar, \hbar - \mu) + \phi(\hbar, \mu) - \phi(\mu, \mu) \geq 0 \quad \forall \hbar \in \mathcal{K} \tag{2}$$

The above formulation is known as the generalized mixed quasi-trifunction variational inequality problem. [11].

2-If $\sup_{\propto \in T\mu} \mathfrak{F}(\mu, \propto, \hbar) \equiv \sup_{\propto \in T\mu} \langle \propto, \hbar - \mu \rangle$, then problem (1) is equivalent to finding $\mu \in \mathcal{K}$ such that

$$\sup_{\propto \in T\mu} \langle \propto, \hbar - \mu \rangle + \sigma(\mu, \hbar) \geq 0 \quad \forall \hbar \in \mathcal{K} \tag{3}$$

This is called as the generalized mixed quasi variational inequality[11] [12] [13] [14] [15] .

Here we write $\sup_{\propto \in T\mu} \langle \propto, \hbar - \mu \rangle$, not $\langle T\mu, \hbar - \mu \rangle$. The reason is simple: T is multivalued ($T: H \to C(H)$). So $T\mu$ is a set, and $\langle T\mu, \cdot \rangle$ has no meaning. We use this same $\sup_{\propto \in T\mu}$ convention throughout the paper.

3- If $\sup_{\propto \in T\mu} \mathfrak{F}(\mu, \propto, \hbar) = \mathfrak{F}(\mu, \hbar)$ then problem (1) is reduced to the generalized equilibrium problem (for short, EP_σ), which is to find $\mu \in \mathcal{K}$ such that $\mathfrak{F}(\mu, \hbar) + \sigma(\mu, \hbar) \geq 0 \ \ \forall \hbar \in \mathcal{K}$;[17].

Lemma 2.1. [10] $\forall \mu, \hbar \in H$

$$2\langle \mu, \hbar \rangle = \|\mu + \hbar\|^2 - \|\mu\|^2 - \|\hbar\|^2. \tag{4}$$

3 Iterative approximation for (NSET)

We now use the (AUXPT) to propose and examine an iterative solution for the issue (1).

For each $\mu \in \mathcal{K}$, suppose the auxiliary equilibrium problem of finding $\mathrm{w} \in H$ such that

$$\rho \sup_{\propto \in T\mu} \mathfrak{F}(\mu, \propto, \hbar) + \langle \mathrm{w} - \mu, \hbar - \mathrm{w} \rangle + \rho\sigma(\mu, \hbar) \geq 0 \quad \forall \hbar \in \mathcal{K} \tag{5}$$

where $\rho > 0$ is a constant. We assume this auxiliary problem has a unique solution w.

if we putting $\mathrm{w} = \mu$, Therefore, w solves of (NSET) (1). This discovery enables us to offer the following iterative strategy for solving problem (1) .

Algorithm 3.1. Given an initial element $\mu_0 \in H$, construct the approximate solution μ_{n+1} by the iterative scheme .

$$\rho \sup_{\mathrm{w} \in T\mathrm{w}_n} F(\mathrm{w}_n, \mathrm{w}, \hbar) + \langle \mu_{n+1} - \mathrm{w}_n, \hbar - \mu_{n+1} \rangle + \rho\sigma(\hbar, \mu_{n+1}) \geq 0 \quad \forall \hbar \in \mathcal{K} \tag{6}$$

$$\beta \sup_{\mathrm{y} \in T\mathrm{y}_n} \mathfrak{F}(\mathrm{y}_n, \mathrm{y}, \hbar) + \langle \mathrm{w}_n - \mathrm{y}_n, \hbar - \mathrm{w}_n \rangle + \beta\sigma(\hbar, \mathrm{w}_n) \geq 0 \quad \forall \hbar \in \mathcal{K} \tag{7}$$

$$\gamma \sup_{\mu \in T\mu_n} \mathfrak{F}(\mu_n, \mu, \hbar) + \langle \mathrm{y}_n - \mu_n, \hbar - \mathrm{y}_n \rangle + \gamma\sigma(\hbar, \mathrm{y}_n) \geq 0 \quad \forall \hbar \in \mathcal{K} \tag{8}$$

In which $\rho > 0$, $\beta > 0$ and $\gamma > 0$ denote positive constants.

The σ-term in (6)–(8) reads $\sigma(\hbar, \cdot)$, with the test point $\hbar$ first. This differs from $\sigma(\cdot, \hbar)$ in the general auxiliary problem (5). We choose this order for a reason: it is exactly what The third step in the proof of Theorem 3.5 needs to obtain the error estimate.

Whenever $\sup_{\propto \in T\mu} \mathfrak{F}(\mu, \propto, \hbar) = \mathfrak{F}(\mu, \hbar)$ then Algorithm (3.1) is reduced to the following iterative procedure.

Algorithm 3.2. Given an initial element $\mu_0 \in H$, construct the approximate solution μ_{n+1} by the iterative scheme .

$$\rho\mathfrak{F}(\mathrm{w}_n, \hbar) + \langle \mu_{n+1} - \mathrm{w}_n, \hbar - \mu_{n+1} \rangle + \rho\sigma(\hbar, \mu_{n+1}) \geq 0 \quad \forall \hbar \in \mathcal{K} \tag{9}$$

$$\beta\mathfrak{F}(\mathrm{y}_n, \hbar) + \langle \mathrm{w}_n - \mathrm{y}_n, \hbar - \mathrm{w}_n \rangle + \beta\sigma(\hbar, \mathrm{w}_n) \geq 0 \quad \forall \hbar \in \mathcal{K} \tag{10}$$

$$\gamma\mathfrak{F}(\mu_n, \hbar) + \langle \mathrm{y}_n - \mu_n, \hbar - \mathrm{y}_n \rangle + \gamma\sigma(\hbar, \mathrm{y}_n) \geq 0 \quad \forall \hbar \in \mathcal{K} \tag{11}$$

In which $\rho > 0$, $\beta > 0$ and $\gamma > 0$ denote positive constants.

if $\sup_{\propto \in T\mu} \mathfrak{F}(\mu, \propto, \hbar) = \sup_{\propto \in T\mu} \langle \propto, \hbar - \mu \rangle$ then Algorithm (3.1) is reduced to the following iterative procedure.

Algorithm 3.3. Given an initial element $\mu_0 \in H$, construct the approximate solution μ_{n+1} by the iterative scheme .

$$\rho \sup_{\propto \in T\mathrm{w}_n} \langle \propto, \hbar - \mathrm{w}_n \rangle + \langle \mu_{n+1} - \mathrm{w}_n, \hbar - \mu_{n+1} \rangle + \rho\sigma(\hbar, \mu_{n+1}) \geq 0 \quad \forall \hbar \in \mathcal{K} \tag{12}$$

$$\beta \sup_{\propto \in T\mathrm{y}_n} \langle \propto, \hbar - \mathrm{y}_n \rangle + \langle \mathrm{w}_n - \mathrm{y}_n, \hbar - \mathrm{w}_n \rangle + \beta\sigma(\hbar, \mathrm{w}_n) \geq 0 \quad \forall \hbar \in \mathcal{K} \tag{13}$$

$$\gamma \sup_{\propto \in T\mu_n} \langle \propto, \hbar - \mu_n \rangle + \langle \mathrm{y}_n - \mu_n, \hbar - \mathrm{y}_n \rangle + \gamma\sigma(\hbar, \mathrm{y}_n) \geq 0 \quad \forall \hbar \in \mathcal{K} \tag{14}$$

As in case 2 of Section 2, we write $\sup_{\propto \in T(\cdot)} \langle \propto, \cdot \rangle$ rather than $\langle T(\cdot), \cdot \rangle$, since T is multivalued.

A three-point monotonicity condition

The convergence proof needs a condition that links $\mathfrak{F}$ at three points: the auxiliary iterate, the solution, and the next iterate. We state this condition below.

Definition 3.4. The trifunction $\mathfrak{F}:\mathcal{K}\times\mathcal{K}\times\mathcal{K}\to R$ is said to be generalized nonstandard relaxed monotone (GNSR) with respect to T , if for all $p, q, r \in \mathcal{K}$,

$$\sup_{\propto\in Tp}\mathfrak{F}(p,\propto,q)+\sup_{y\in Tq}\mathfrak{F}(q,y,r)+\sigma(q,r)\le 0. \tag{15}$$

By selecting appropriate operators and an appropriate Hilbert space H, Algorithm (3.1) can be specialized to recover several existing methods as well as derive new iterative schemes for variational inequalities and complementarity problems. The following theorem is required to establish the convergence of Algorithm (3.1).

Theorem 3.5. suppose that $\overline{\mu}\in\mathcal{K}$ is a solution of problem (1) and the sequences μ_{n+1}, w_n, y_n are generated by Algorithm (3.1). If $F:\mathcal{K}\times\mathcal{K}\times\mathcal{K}\to R$ is (GNSR) with respect to T (Definition 3.4), then:

$$\|\mu_{n+1}-\overline{\mu}\|^2\le 2\rho\sigma(\overline{\mu},\mu_{n+1})-\|\mu_{n+1}-w_n\|^2+\|w_n-\overline{\mu}\|^2 \tag{16}$$

$$\|w_n-\overline{\mu}\|^2\le 2\beta\sigma(\overline{\mu},w_n)-\|w_n-y_n\|^2+\|y_n-\overline{\mu}\|^2 \tag{17}$$

$$\|y_n-\overline{\mu}\|^2\le 2\gamma\sigma(\overline{\mu},y_n)-\|y_n-\mu_n\|^2+\|\mu_n-\overline{\mu}\|^2 \tag{18}$$

Proof. We prove (16) in full detail; (17) and (18) follow by the identical argument under the index shift $(\mu_{n+1}, w_n, y_n)\to(w_n, y_n, \mu_n)\to(y_n, \mu_n, \mu_{n+1})$ indicated at the end of the proof.

Step 1: Since $\overline{\mu}\in\mathcal{K}$ solves (1), taking $\hbar=\mu_{n+1}$ in (1) gives

$$\sup_{x\in T\overline{\mu}}\mathfrak{F}(\overline{\mu},\propto,\mu_{n+1})\ge-\sigma(\overline{\mu},\mu_{n+1}). \tag{19}$$

Step 2 : Apply Definition 3.4 with $p=w_n$, $q=\overline{\mu}$, $r=\mu_{n+1}$:

$$\sup_{x\in Tw_n}\mathfrak{F}(w_n,\propto,\overline{\mu})+\sup_{y\in T\overline{\mu}}\mathfrak{F}(\overline{\mu},y,\mu_{n+1})+\sigma(\overline{\mu},\mu_{n+1})\le 0.$$

Rearranging and substituting the bound (19),

$$\sup_{x\in Tw_n}\mathfrak{F}(w_n,\propto,\overline{\mu})\le-\sup_{y\in T\overline{\mu}}\mathfrak{F}(\overline{\mu},y,\mu_{n+1})-\sigma(\overline{\mu},\mu_{n+1})\le\sigma(\overline{\mu},\mu_{n+1})-\sigma(\overline{\mu},\mu_{n+1})=0. \tag{20}$$

This bound follows from the solution property (19) and Definition 3.4.

Step 3 : Set $\hbar=\overline{\mu}$ in (6):

$$\rho\sup_{w\in Tw_n}\mathfrak{F}(w_n,w,\overline{\mu})+\langle\mu_{n+1}-w_n,\overline{\mu}-\mu_{n+1}\rangle+\rho\sigma(\overline{\mu},\mu_{n+1})\ge 0.$$

By (20) the first term is ≤ 0, so

$$\langle\mu_{n+1}-w_n,\overline{\mu}-\mu_{n+1}\rangle\ge-\rho\sup_{w\in Tw_n}\mathfrak{F}(w_n,w,\overline{\mu})-\rho\sigma(\overline{\mu},\mu_{n+1})\ge-\rho\sigma(\overline{\mu},\mu_{n+1}). \tag{21}$$

Step 4 : Applying Lemma 2.1 with $\mu=\mu_{n+1}-w_n$ and $\hbar=\overline{\mu}-\mu_{n+1}$,

$$2\langle\mu_{n+1}-w_n,\overline{\mu}-\mu_{n+1}\rangle=\|w_n-\overline{\mu}\|^2-\|\overline{\mu}-\mu_{n+1}\|^2-\|\mu_{n+1}-w_n\|^2. \tag{22}$$

Multiplying (21) by 2 and substituting the left-hand side using (22),

$$\|w_n-\overline{\mu}\|^2-\|\overline{\mu}-\mu_{n+1}\|^2-\|\mu_{n+1}-w_n\|^2\ge-2\rho\sigma(\overline{\mu},\mu_{n+1}),$$

which rearranges to exactly (16):

$$\|\mu_{n+1}-\overline{\mu}\|^2\le 2\rho\sigma(\overline{\mu},\mu_{n+1})-\|\mu_{n+1}-w_n\|^2+\|w_n-\overline{\mu}\|^2. \tag{23}$$

Steps 1–4 for (17) and (18). For (17), repeat Steps 1–4 verbatim with the substitutions $\mu_{n+1}\to w_n$, $w_n\to y_n$, using (7) in place of (6) and Definition 3.4 with $(p,q,r)=(y_n,\overline{\mu},w_n)$ in place of Step 2. For (18), repeat with $\mu_{n+1}\to y_n$, $w_n\to\mu_n$, using (8) in place of (6) and Definition 3.4 with $(p,q,r)=(\mu_n,\overline{\mu},y_n)$. In each case the intermediate bound analogous to (20) is

$$\sup_{x\in Ty_n}\mathfrak{F}(y_n,\propto,\overline{\mu})\le 0\qquad\text{and}\qquad\sup_{x\in T\mu_n}\mathfrak{F}(\mu_n,\propto,\overline{\mu})\le 0$$

respectively, and the remainder of the argument is identical. ▫

Corollary 3.6. Throughout this corollary, we assume T is an upper semicontinuous set-valued mapping on $\mathcal{K}$ with compact images, and $\mathfrak{F}(\cdot,\cdot,\hbar)$ is continuous in its first two arguments for each fixed $\hbar$. Let H be finite dimensional, and suppose problem (1) has a solution. Suppose that for every solution $\overline{\hbar}\in\mathcal{K}$ of problem (1),

$$\sigma(\overline{\hbar},\hbar)\le 0\qquad\text{for every }\hbar\in\{\mu_n, w_n, y_n : n\ge 0\},$$

where $\{\mu_n\}$, $\{w_n\}$, $\{y_n\}$ are the sequences produced by Algorithm (3.1). Then $\mu_n\to\hat{\mu}$ for some solution $\hat{\mu}$ of problem (1).

Remark 3.7. The upper semicontinuity hypothesis in Corollary 3.6 is used once, to pass to the limit inside $\sup_{\propto\in T(\cdot)}\mathfrak{F}(\cdot,\propto,\hbar)$: it gives $\mu_{n_i}\to\hat{\mu}$ implies $\sup_{\propto\in T\mu_{n_i}}\mathfrak{F}(\mu_{n_i},\propto,\hbar)\to\sup_{\propto\in T\hat{\mu}}\mathfrak{F}(\hat{\mu},\propto,\hbar)$. Compact values of T alone do not give this; it must be assumed separately.

Proof. Fix any solution $\overline{\mu}$ of problem (1), and write $a_n=\|\mu_n-\overline{\mu}\|^2$. Since $\sigma(\overline{\mu},\cdot)\le 0$ on the generated sequences, (16)–(18) give, for every n,

$$a_{n+1}\le\|w_n-\overline{\mu}\|^2\le\|y_n-\overline{\mu}\|^2\le a_n. \tag{24}$$

So $\{a_n\}$ is non increasing, hence bounded and convergent, say $a_n\to L\ge 0$; in particular $\{\mu_n\}$ is bounded, and by (24) so are $\{w_n\}$ and $\{y_n\}$.

Next we bound the increments between consecutive iterates. Rearranging (16) and using $\sigma(\overline{\mu}, \mu_{n+1}) \leq 0$,

$$\| \mu_{n+1} - \mathrm{w}_n \|^2 \leq \| \mathrm{w}_n - \overline{\mu} \|^2 - a_{n+1} \leq a_n - a_{n+1},$$

using $\| \mathrm{w}_n - \overline{\mu} \|^2 \leq a_n$ from (24). In the same way, (17) and (18) give

$$\| \mathrm{w}_n - \mathrm{y}_n \|^2 \leq \| \mathrm{y}_n - \overline{\mu} \|^2 - \| \mathrm{w}_n - \overline{\mu} \|^2 \leq a_n - a_{n+1}, \qquad \| \mathrm{y}_n - \mu_n \|^2 \leq a_n - \| \mathrm{y}_n - \overline{\mu} \|^2 \leq a_n - a_{n+1},$$

again using (24) to bound $\| \mathrm{y}_n - \overline{\mu} \|^2$ between a_{n+1} and a_n. Summing each of these three bounds from $n = 0$ to N telescopes the right-hand side to $a_0 - a_{n+1} \leq a_0 - L$, so

$$\sum_{n=0}^{\infty} \| \mu_{n+1} - \mathrm{w}_n \|^2 < \infty, \qquad \sum_{n=0}^{\infty} \| \mathrm{w}_n - \mathrm{y}_n \|^2 < \infty, \qquad \sum_{n=0}^{\infty} \| \mathrm{y}_n - \mu_n \|^2 < \infty,$$

and consequently

$$\lim_{n\to\infty} \| \mu_{n+1} - \mathrm{w}_n \| = \lim_{n\to\infty} \| \mathrm{w}_n - \mathrm{y}_n \| = \lim_{n\to\infty} \| \mathrm{y}_n - \mu_n \| = 0. \tag{25}$$

By the triangle inequality, (25) gives $\| \mu_n - \mathrm{w}_n \| \to 0$ and $\| \mu_n - \mathrm{y}_n \| \to 0$ as well: the three sequences become arbitrarily close to each other as $n \to \infty$.

Since H is finite dimensional and $\{\mu_n\}$ is bounded, by Bolzano–Weierstrass theorem (Every bounded sequence in a finite-dimensional normed space admits a convergent subsequence),there exists a subsequence $\mu_{n_i} \to \hat{\mu}$ for some $\hat{\mu} \in \mathcal{K}$. Because $\| \mu_{n_i} - \mathrm{w}_{n_i} \| \to 0$ and $\| \mu_{n_i} - \mathrm{y}_{n_i} \| \to 0$, the same subsequence gives $\mathrm{w}_{n_i} \to \hat{\mu}$ and $\mathrm{y}_{n_i} \to \hat{\mu}$. Taking the limit $n_i \to \infty$ in (6), (7), (8), and using the upper semicontinuity assumption on $\mathfrak{F}$ and T recorded in Remark 3.7, we get

$$\sup_{x \in T\hat{\mu}} \mathfrak{F}(\hat{\mu}, \propto, \hbar) + \sigma(\hat{\mu}, \hbar) \geq 0 \qquad \forall \hbar \in \mathcal{K},$$

We conclude that $\hat{\mu}$ is a solution to problem (1).

It remains to show the whole sequence $\{\mu_n\}$, not just this subsequence, converges to $\hat{\mu}$. Since $\hat{\mu}$ is itself a solution of problem (1), the hypothesis gives $\sigma(\hat{\mu}, \cdot) \leq 0$ on the generated sequences, so the argument leading to (24) applies with $\overline{\mu}$ replaced by $\hat{\mu}$ the sequence $\tilde{a}_n = \| \mu_n - \hat{\mu} \|^2$ is non increasing. Since $\mu_{n_i} \to \hat{\mu}$, we have $\tilde{a}_{n_i} \to 0$, and a non increasing sequence with a subsequence converging to 0 converges to 0 itself. Hence $\tilde{a}_n \to 0$, that is, $\mu_n \to \hat{\mu}$. ▫

4 Conclusion

We introduced the σ-nonstandard equilibrium trifunction variational inequality, a class of problems built on a multivalued operator T. We gave a three-step iterative algorithm for this problem, based on the auxiliary principle technique. We stated a three-point monotonicity condition on F (Definition 3.4) and used it to give a full, step-by-step proof of the main error estimate (Theorem 3.5). Under this condition, together with a sign condition on σ along the generated sequences and an upper semicontinuity hypothesis (Remark 3.7), we obtained convergence. We showed the whole sequence of iterates converges to a solution (Corollary 3.6). The proposed framework encompasses several well-established classes of variational inequalities and equilibrium problems as particular cases.

References

[1] G. Stampacchia, Formes bilinéaires coercitives sur les ensembles convexes, *Académie des Sciences de Paris*, 258 (1964), 4413–4416.

[2] B. Martinet, Régularisation d'inéquations variationnelles par approximation successive, *Revue d'Automatique, Informatique, et Recherche Opérationnelle*, 3 (1970), 154–159.

[3] U. Mosco, Implicit variational problems and quasi variational inequalities, *Lecture Notes in Mathematics*, 543 (1976), 83–126.

[4] R. Glowinski, J. Lions, R. Trémolières, *Numerical Analysis of Variational Inequalities*, North-Holland, Amsterdam, 1981.

[5] M. A. Noor, Nonlinear variational inequalities in elastostatics, *International Journal of Engineering Sciences*, 26 (1988), 1043–1053.

[6] M. A. Noor, Mixed quasi variational inequalities, *Applied Mathematics and Computations*, 146 (2003), 553–578.

[7] M. A. Noor, Some developments in general variational inequalities, *Applied Mathematics and Computation*, 152 (2004), 199–277.

[8] M. A. Noor, Implicit iterative methods for nonconvex variational inequalities, *J. Optim. Theory Appl.*, 143 (2009), 619–624.

[9] M. A. Noor, On an implicit method for nonconvex variational inequalities, *J. Optim. Theory Appl.*, 147 (2010).

[10] Netra Kumar Gupta, Suja Varghese, M.A. Siddiqui, Muhammad Aslam Noor, General mixed quasi equilibrium variational inequalities, *Global Journal of Pure and Applied Mathematics*, 21(3) (2025), 407–418.

[11] Muhammad Aslam Noor, Generalized mixed quasi trifunction variational inequalities, *Journal of King Saud University – Science*, 23 (2011), 171–174.

[12] F. Giannessi, A. Maugeri, P.M. Pardalos, *Equilibrium Problems: Nonsmooth Optimization and Variational Inequality Models*, Kluwer Academic Publishers, 2001.

[13] A. Maugeri, F. Giannessi, *Variational Inequalities and Network Equilibrium Problems*, Plenum Press, New York, 1995.

[14] F. Flores-Bazán, Existence theorems for generalized noncoercive equilibrium problems: the quasi-convex case, *SIAM Journal on Optimization*, 11 (2000), 675–690.

[15] R.T. Rockafellar, Monotone operators and the proximal point algorithm, *SIAM Journal on Control and Optimization*, 14 (1976), 877–898.

[16] E. Blum, W. Oettli, From optimization and variational inequalities to equilibrium problems, *The Mathematics Student*, 63 (1994), 123–145.

[17] Ayed E. Hashoosh, Mohsen Alimohammady, Morteza K. Kalleji, Existence results for some equilibrium problems involving α-monotone bifunction, *International Journal of Mathematics and Mathematical Sciences*, 2016 (2016), 2093026.

[18] W. S. Abed, A. E. Hashoosh, D. M. Abbas, Existence results for some equilibrium problems involving B_h-monotone trifunction, *AIP Conf. Proc.*, 2414 (2023), 040015. https://doi.org/10.1063/5.0136137.

[19] H. M. B. Alrikabi, A. E. Hashoosh, A. A. H. Alkhalidi, Existence results for nonlocal problems of (p, q)-Kirchhoff form, *J. Interdiscip. Math.*, 24 (2021), 983–993.

[20] A. A. Alwan, H. A. Naser, A. E. Hashoosh, The solutions of mixed hemiequilibrium problem with application in Sobolev space, *IOP Conf. Ser. Mater. Sci. Eng.*, 571 (2019), 012019. https://doi.org/10.1088/1757-899X/571/1/012019.

[21] A. E. Hashoosh, Existence and uniqueness solutions for a class of hemivariational inequalities, *J. Math. Inequal.*, 11 (2017), 565–576. https://doi.org/10.7153/jmi-11-46.

[22] A. E. Hashoosh, A. M. Ali, A modern method for identifying fixed points used in $\mathcal{G}$-spaces involving application, *AIP Conf. Proc.*, 3264 (2025), 050050.

[23] A. E. Hashoosh, A. M. Jebur, Improvement of well-posedness for hemiequilibrium problems, *Math. Eng. Sci. Aerosp.*, 12 (2021), 5–17.