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Power Domination And Related Variants In Graphs: A Survey And Future Directions

R. Bhavani[1], S. Banu Priya[2*]

[1]Department of Mathematics, St. Peter's Institute of Higher Education and Research, AVADI, Chennai-600054, Email: bhavani.maths3001@gmail.com
[2*] Department of Mathematics, Velammal Institute of Technology, Email: banupriyasadagopan@gmail.com

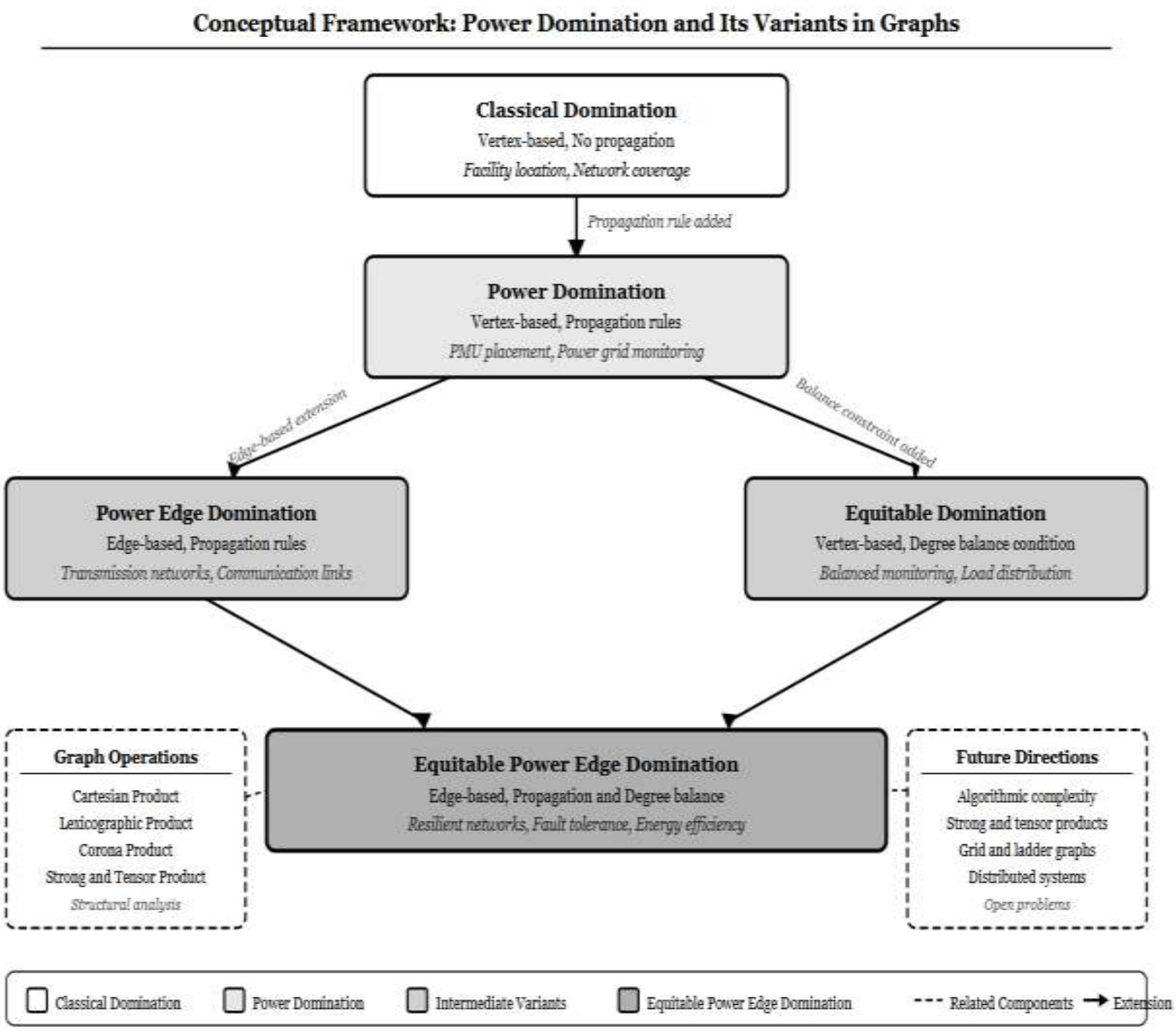
Abstract
Power domination is a major extension of the classical domination in graphs inspired by the problem of monitoring an electric power system by "phasor measurement units". In comparison with the classical domination, power domination includes some propagation rules that enable a small number of measuring units to observe the entire power network. There are many different variations and extensions of power domination, such as power edge domination, equitable power edge domination, etc., that were proposed to meet specific requirements of the power system and related structures. The goal of this article is to present a working survey on power domination, which would give a reader a comprehensive understanding of this concept and its variations. We focus our analysis on the results related to the power domination on graphs, including product of graphs, and power monitoring networks and propose a number of open problems on domination in graphs in general.

Keywords: Power domination, Power edge domination, Equitable power domination, Graph domination, Graph products, Network monitoring, Survey, Future directions.

1. Introduction

Domination theory plays an important role in graph theory and has been studied extensively in recent decades because of its applications to network monitoring, facility location, and other fields. In the classical domination problem, a minimum set of vertices or edges that are sufficient to monitor the whole graph with a given graph structure is identified. However, in some cases, the monitoring effect of vertices or edges will spread to other vertices or edges, and many graph theorists put forward some domination parameters related to the propagation of monitor based on the classical domination. Power domination was introduced for the first time to model the placement of monitoring devices in an electrical power system, where the monitoring of the system is achieved by placing monitoring devices in certain positions and propagating measurements through the network. It has been widely studied on various graphs and networks, and power domination has been investigated extensively for many graph operations since it was proposed. To fully reflect fairness, connectivity, and balance, power edge domination and equitable domination were proposed. In this article, we summarize the results of power domination and its variations in graphs comprehensively. We present some definitions, theoretical results, and applications, with emphasis on the graph products, and other operations. In addition, we also propose some open problems and further study directions, which will provide references for further study in this direction in the future. The remainder of this paper is organized as follows: Section 2 introduces the basic concepts and definitions. Section 3 presents the literature review. Section 4 shows the application of power domination. Section 5 lists the open problems and future research directions and finally Section 6 concludes this paper.

Figure 1: Conceptual framework illustrating the hierarchical relationship between Power Domination and its variants in graphs.



2. Basic Concepts and Terminology

All graphs used in this survey are finite, simple, and connected. Let G=(V(G),E(G)) be a graph with vertex set V(G) and edge set E(G). The degree of a vertex v is denoted by deg(v). A dominating set of a graph G is defined as a a subset D of the vertex set V(G)such that any vertex not in D is adjacent to at least one vertex in D. The minimum cardinality of a dominating set is called the domination number γ(G) of G. Another variant of domination is Power domination, which is a concept derived from the study of monitoring electrical networks. A power dominating set of vertices such that all vertices of a graph are made visible through domination and propagation. A variant of domination is power edge domination, where instead of dominating vertices, edges are monitored. Equitable domination is a variant of domination where balance is a constraint on domination.

3. Literature Review on Power Domination and Its Variants

	Year	Topic / Parameter	Main Contribution
Brueni and Heath	2005	Power domination	Introduced PMU placement model and propagation rules
Dorbec et al.	2008	Power domination in product graphs	Studied behavior under graph products

Klavžar and Zmazek	2005	Domination in Cartesian products	Structural results and bounds
Klavžar and Seifter	1995	Domination in product graphs	Results for cycles and product graphs
Xu and Yang	2013	Connectivity in product graphs	Structural properties of lexicographic products
Martínez and Rodríguez-Velázquez	2022	Protection parameters	Results in lexicographic product graphs

"Power domination" was introduced as a model of the location of "phasor measurement units" in power networks. The idea was first formalized in a paper on the "PMU placement problem" by Brueni and Heath. They analyzed the problem of "monitoring" in the network using a set of "simple observation rules." Since this paper, the concept of power domination has been extensively analyzed in a number of graph families. The primary focus of the research has been to calculate the power domination numbers for different types of graphs. The research has also focused on the concept of power domination with respect to various graph operations, such as the Cartesian product, lexicographic product, and corona product, which represent large networks in the form of smaller networks. In order to address the other constraints, various types of power domination have been proposed. One such variant of power domination is the concept of power edge domination, in which the monitoring is performed on the edges of the graph, unlike the vertices. This concept is more important in the context of network environment, where the primary component is the communication links/transmission lines. Another major variation of power domination is equitable domination, where the balance constraint has been added to ensure equitable domination among the selected elements of the graph. When the propagation rules and the balance constraint are added, the parameters of equitable power domination and power edge domination are achieved. Investigations have also been carried out to study the parameters of domination in the context of graph products and complex networks. These research investigations provide an understanding of the variations when smaller networks are combined, which includes some open issues in terms of computational complexity.

Table 1: Comparative Analysis of Power Domination and Equitable Variants

Graph Type(G)	Power Domination $\gamma_p(G)$	Power Edge Domination $\gamma_{pe}(G)$	Equitable Power Edge Domination $\gamma'_{eped}(G)$
Path (P_n)	1	1	$\lceil n/3 \rceil$
Cycle (C_n)	1	1	$\lceil n/3 \rceil$
Star ($K_{1,n}$)	1	1	1
Complete(K_n)	1	$\lceil n/3 \rceil$	$\lceil (n-1)/2 \rceil$
Wheel (W_n)	1	1	$\lceil n/4 \rceil$

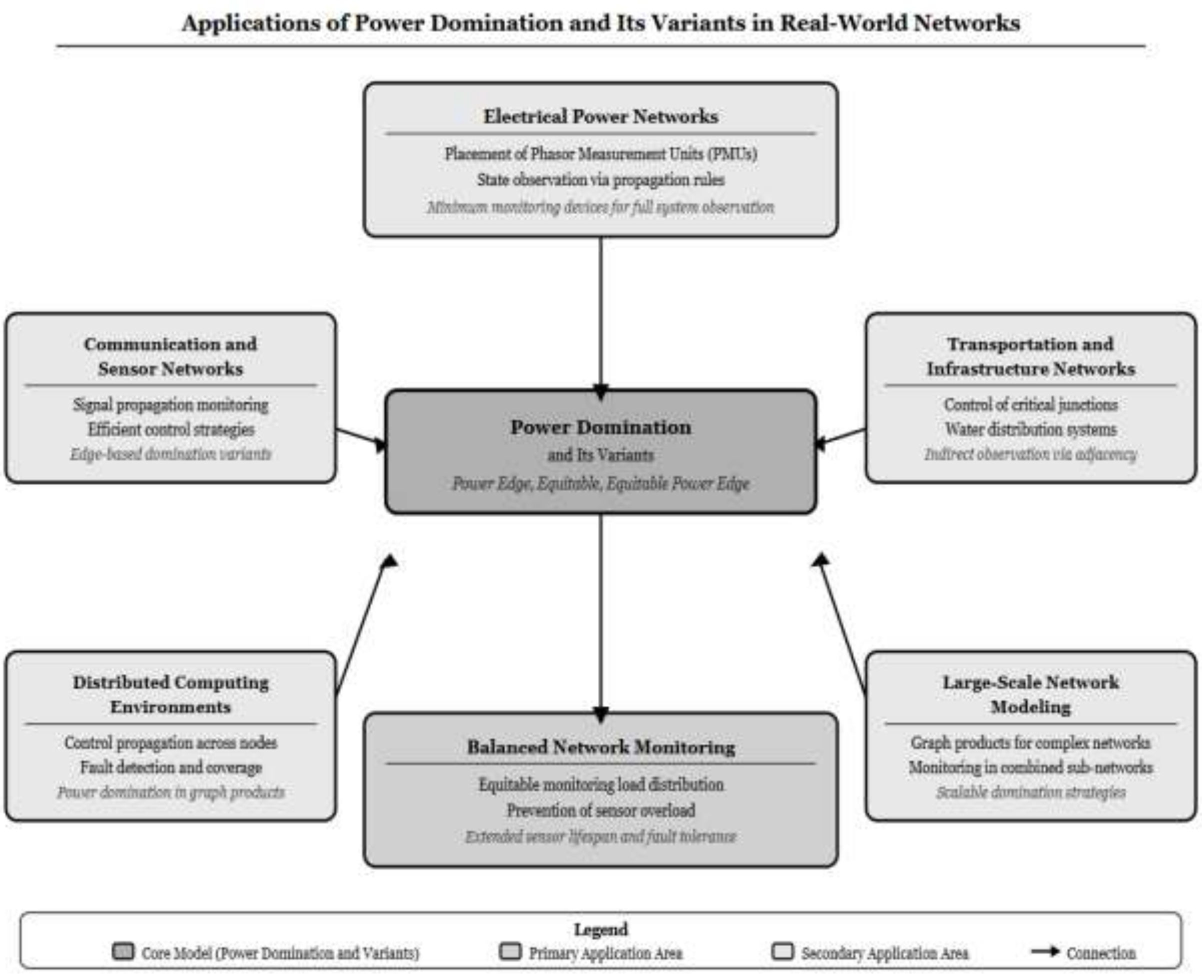
Table 2: Comparison of Domination Variants in Graphs

Parameter	Monitoring Unit	Propagation Rule	Equitable Condition	Typical Applications
Domination	Vertices	No propagation	Not required	Facility location, network coverage
Power Domination	Vertices	Yes (propagation rules)	Not required	Power grid monitoring, PMU placement
Power Edge Domination	Edges	Yes	Not required	Communication links, transmission networks
Equitable Domination	Vertices	No propagation	Degree difference ≤ 1	Balanced monitoring
Equitable Power Edge Domination	Edges	Yes	Degree difference ≤ 1	Balanced network monitoring

4. Applications of Power Domination

Power domination and its variations have many applications in networks of practical interest. The initial motivation came from the problem of monitoring the electric power system, where Phasor Measurement Units (PMUs) are placed at particular buses to observe the state of the electric power grid using measurements from these units and the governing laws of the electric circuit (Ohm's and Kirchhoff's laws); graph theory provides the foundation for determining the minimum number of PMUs needed for full observability.
However, monitoring issues arise in other types of networks, such as communication and wireless sensor networks, where signals propagate through communication channels. This makes the edge-based variations of power domination particularly valuable, since communication links may be considered as edges in a graph. The same justification applies to various other networks, including transportation (roads), water distribution, and distributed computing networks. Additionally, the requirement of equitable domination is essential in many cases, as it prevents any single node from being overwhelmed by the excessive traffic, thus maximizing the network's performance, and ensuring the longevity of the equipment (in particular, batteries in sensor networks).

Figure 2: Applications of Power Domination and its variants in real-world network systems.



Another important domain for the application of the power domination problem is in communication/sensor networks, where the spread of monitoring signals from node to node along communication channels may be represented, and domination-based methods may be helpful in designing efficient monitoring strategies. The edge-based power domination problem may be useful in situations where communication links or channels are the main components being monitored. The power domination concepts may be useful in transportation networks, water supply networks, and distributed computing, where the monitoring of some components may serve as an indirect monitoring of other components. The equitable power domination may be useful in situations where the occurrence of any overload situation in the network needs to be prevented. The power domination graph product may be useful in situations where it is necessary to model a large network constructed from smaller sub-networks. The real-world practicality of the Equitable Power Edge Domination Number $\gamma'_{eped}(G)$ is best illustrated in the context of the "cost of fairness" in high-density networks. Although the conventional power domination number $\gamma_p(G)$ attempts to achieve the absolute minimum number of sensor nodes, there exists a "bottleneck" in the conventional approach wherein a single node is responsible for a very large section of the network. As can be seen in Table 1, in the case of the Complete Graph(K_n), the power domination number is always equal to 1, implying that one vertex is responsible for the entire monitoring. On the other hand, $\gamma'_{eped}(G)$ demands $\lceil (n-1)/2 \rceil$ edges for an equitable allocation. This is done as a deliberate measure to avoid hardware burnout, minimize data processing latency, and ensure that no single point of failure can blindside a large portion of the network. However, the Equitable Power Edge Domination Number $\gamma'_{eped}(G)$ requires at least $\lceil (n-1)/2 \rceil$ edges. This is done as a deliberate step to avoid burnout of the hardware, as well as the latency associated with the processing of the data, and also to avoid situations where a single point of failure blindsides a huge part of the network.

4.1 Advanced Applications of Equitable Monitoring

The introduction of equitable constraints into power edge domination provides several advantages for the design and maintenance of large-scale complex networks.

Dynamic Load Balancing and Scalability

In the context of modular data centers, the network can often be built by scaling up smaller identical networks through graph products such as the Cartesian product. The traditional power domination model can potentially place a high monitoring load on any given vertex, referred to as the "gateway," as the network is expanded. However, through the application of equitable constraints, it is possible to ensure that the increased monitoring load is automatically distributed across the expanded infrastructure.

Optimization of Sensor Lifespan in WSNs

For Wireless Sensor Networks (WSNs), the main constraint for operation is energy consumption. If a node is assigned the duty of observing a large neighborhood, which is common for Minimum Power Domination Sets, the energy depletion of this node will occur much faster than those of the other nodes. An equal distribution will prevent this from occurring. A larger set of edges ($\gamma'_{eped}(G)$) will spread the observation duty uniformly, resulting in an equal energy depletion rate for all the nodes and increasing the overall lifespan of the system.

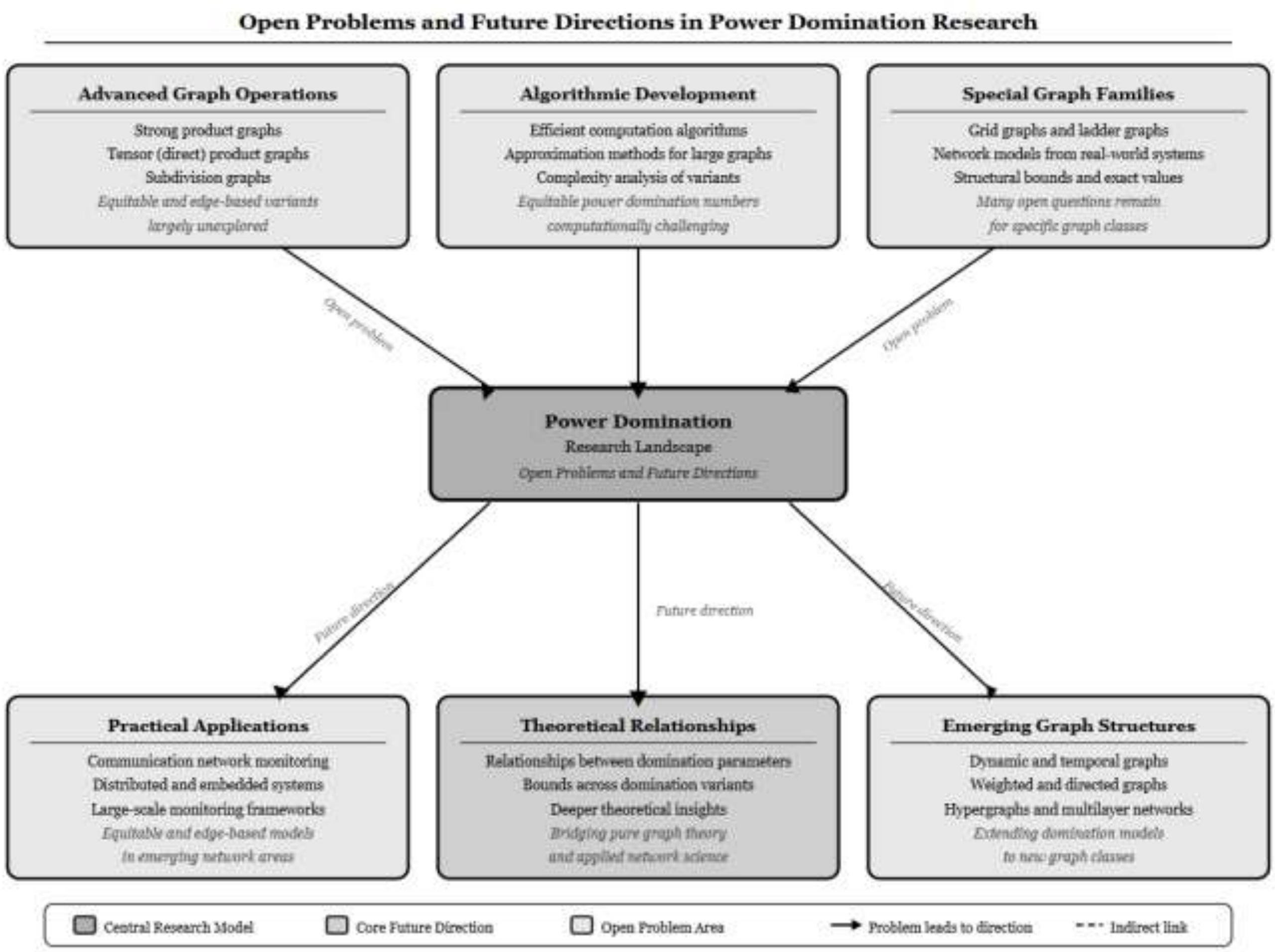
Enhanced Fault Tolerance and Redundancy

While a higher value for $\gamma'_{eped}(G)$ relative to $\gamma_p(G)$ might seem less "efficient" in terms of cost, it significantly improves system redundancy. In a single-sensor power domination model, the failure of that one node results in a total loss of observability for the entire network. Conversely, the "fair" distribution inherent in an equitable set ensures that if one monitoring unit fails, the surrounding nodes are only slightly more burdened to maintain coverage. This inherent redundancy provides a robust safety net for critical infrastructure such as electrical grids and Even though a higher value for $\gamma'_{eped}(G)$ compared to $\gamma_p(G)$ might be less "efficient" in terms of costs, it greatly enhances the redundancy factor. In a single sensor power domination model, failure of that particular sensor results in a total lack of observability for the entire network. However, with the "fair" factor inherent in an equitable set, when a monitoring unit fails, the surrounding units are only slightly more burdened to ensure total coverage. This redundancy factor acts as a safety net for critical infrastructure systems, such as the electrical and communication grids communication channels.

5. Open Problems and Future Directions

Although the study of power domination and its variations is growing and promising, there are still interesting problems to be explored. First, the power domination related parameters of more complicated graph operations, such as the strong product, tensor (direct) product, subdivision graphs, etc., and their varied forms (including Cartesian product, lexicographic product), are waiting to be studied. Although some works have been done on the power domination of graph products, many other power domination variants and graph products have not been studied yet, not to mention their equitable or edge versions.

Figure 3: Open problems and future research directions in Power Domination and its variants.



Another interesting problem is the development of efficient algorithms for calculating power domination and equitable power domination numbers. It is proved that the calculation of these numbers is a difficult task even for large graphs, so the development of algorithms that allow their approximate calculation is also of interest. It is also of interest to study the properties of graphs that allow the exact calculation of these numbers and give bounds for these numbers on various graph classes. The most promising graphs for the study are grids, ladder graphs, and graphs representing some real networks. In addition, it is of applied interest to study domination in graphs for application in communication and transport networks, modeling various processes with monitoring and control systems, and developing recommendations for solving power domination problems in specific networks. Of particular interest is the study of the relationships between different domination numbers.

Conclusion

In this survey article, we give a brief background about power domination and related concepts like power edge domination and equitable domination with an overview of the most important results related to these problems in the literature. We discuss the effect of various graph operations on monitoring problems. We present the motivation to study the concept of equitable power edge domination and explain why it is needed, besides classical domination concepts, to design and maintain real-world networks where devices or sensors have limited or finite lifetime. We also highlight some challenges related to strong products and give a direction for future studies on approximation algorithms. The article is written to provide a vision of the domination theory area for researchers interested in studying variants and applications of domination in graphs.

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