https://doi.org/10.51483/IJPAMR.4.2.2024.63-69 ISSN: 2789-9160

SvedbergOpen
DISSEMINATION OF KNOWLEDGE

International Journal of Pure and Applied Mathematics Research

Publisher's Home Page: https://www.svedbergopen.com/

Research Paper Open Access

Recursive Formula for Sum of Powers of Natural Numbers and its Generalization to Arithmetic Progression

Ahmad Daffa I.F. Askari[1*]

[1]Faculty of Mathematics and Natural Sciences, Gadjah Mada University, Indonesia. E-mail: adffask16@gmail.com

Article Info

Volume 4, Issue 2, October 2024
Received : 06 June 2024
Accepted : 21 September 2024
Published : 05 October 2024
doi: 10.51483/IJPAMR.4.2.2024.63-69

Abstract

In this paper, we derive a formula for sum of powers of integers from Abel's Summation Formula. This formula enables us to generate the formula for the sum of k^{th} power of integers, denoted by $S_k(n)$, given the formulas of $S_1(n)$, $S_2(n)$, ..., $S_{k-1}(n)$. Furthermore, we shall extend this formula to compute the sum of powers of an arithmetic progression. Moreover, we can combine the formula with the result of Bernoulli (Beardon, 1996) to derive another result which enables us to find Bernoulli Numbers recursively.

Keywords: *Sum of powers, Arithmetic progression, Bernoulli numbers*

1. Notation

For the sake of convenience, we shall denote the sum of powers of the first n natural numbers as follows

$$S_m(n) = \sum_{i=1}^{n} i^m$$

As we shall see in the following sections, the notation will reduce the need to write the sum notation which could be cumbersome to work with.

2. The Evaluation of Special Kind of Integral Involving Floor Function

As we will see in the proof of the recursive formula, the proof will involve the evaluation of the following integral.

$$\int_1^n t^\alpha \lfloor t \rfloor dt$$

where α and n are positive integers

The integral above cannot be evaluated by using the regular approach, namely, by the product rule, since the function $f(x) = \lfloor x \rfloor$ has discontinuities. Thus, in order to evaluate the integral, there is one important property of floor

* *Corresponding author: Ahmad Daffa I.F. Askari, Faculty of Mathematics and Natural Sciences, Gadjah Mada University, Indonesia. E-mail: adffask16@gmail.com*

function that we must consider. That is, $\lfloor t \rfloor = k$ for all $t \in [k, k+1)$. The property then motivates us to use another approach to calculate the area under the curve of the integrand that we have above. The idea is to calculate separately the area under the curve on interval $[k, k+1)$ for $k = 1, 2, ..., n-1$ then we sum them up. Also note that if k is any integer that we choose from $\{1, 2, ..., n-1\}$ then for all t in the interval $[k, k+1)$, $\lfloor t \rfloor = k$ which is a constant.

Lemma 2.1.: For the integral above, we have

$$\int_1^n t^\alpha \lfloor t \rfloor dt = \frac{1}{\alpha+1}\sum_{i=1}^{\alpha+1}\binom{\alpha+1}{i-1}S_{i-1}(n-1)$$

Proof: For the sake of convenience, let us denote the area of the integrand on $[k, k+1)$ as I_k. More precisely,

$$I_k = \int_k^{k+1} t^\alpha \lfloor t \rfloor dt$$

Recall that $\lfloor t \rfloor = k$ for all $t \in [k, k+1)$ so we can take it out of the integral. Now,

$$I_k = k\int_k^{k+1} t^\alpha dt$$

$$= k\frac{(k+1)^{\alpha+1} - k^{\alpha+1}}{\alpha+1}$$

Therefore, if I denotes the integral that we are evaluating, then

$$I = \sum_{k=1}^{n-1} I_k$$

$$= \sum_{k=1}^{n-1} k\frac{(k+1)^{\alpha+1} - k^{\alpha+1}}{\alpha+1}$$

$$= \frac{1}{\alpha+1}\sum_{k=1}^{n-1}\left(k\left((k+1)^{\alpha+1} - k^{\alpha+1}\right)\right) \qquad ...(1)$$

By Binomial Theorem, now we have

$$(k+1)^{\alpha+1} = \sum_{i=0}^{\alpha+1}\binom{\alpha+1}{i}k^i$$

$$= \sum_{i=1}^{\alpha+2}\binom{\alpha+1}{i-1}k^{i-1} \qquad ...(2)$$

Substituting (2) to (1), we have

$$I = \frac{1}{\alpha+1}\sum_{k=1}^{n-1}\left(\sum_{i=1}^{\alpha+2}\binom{\alpha+1}{i-1}k^{i-1} - k^{a+1}\right)$$

$$= \frac{1}{\alpha+1}\sum_{k=1}^{n-1}\left(k^{\alpha+1} + \sum_{i=1}^{\alpha+1}\binom{\alpha+1}{i-1}k^{i-1} - k^{a+1}\right)$$

$$= \frac{1}{\alpha+1}\sum_{k=1}^{n-1}\left(\sum_{i=1}^{\alpha+1}\binom{\alpha+1}{i-1}k^{i-1}\right)$$

Interchanging the sum and applying our notation, now we find that

$$I = \frac{1}{\alpha+1}\sum_{i=1}^{\alpha+1}\left(\sum_{k=1}^{n-1}\binom{\alpha+1}{i-1}k^{i-1}\right) = \frac{1}{\alpha+1}\sum_{i=1}^{\alpha+1}\binom{\alpha+1}{i-1}S_{i-1}(n-1)$$

3. Main Results

3.1. Recursive Formula of Sum of Powers and Bernoulli Numbers

Theorem 3.1.: Let $S_m(n) = \sum_{i=1}^{n} i^m$ then,

$$S_m(n) = \frac{1}{m+1}\left(n(n+1)^m - \sum_{i=1}^{m-1}\binom{m}{i-1}S_i(n)\right)$$

Proof: Let us begin with Abel's Summation Formula (Tom, 2013). We assume that n is a positive integer. Hence, $\lfloor n \rfloor = n$

$$\sum_{i=1}^{n} f(i) = nf(n) - \int_1^n f'(t)\lfloor t \rfloor dt$$

Let $f(i) = i^m$

$$\sum_{i=1}^{n} i^m = n \cdot n^m - \int_1^n mt^{m-1}\lfloor t \rfloor dt = n^{m+1} - m\int_1^n t^{m-1}\lfloor t \rfloor dt$$

The integral above is the special case of the integral that we have solved in the previous section shown in lemma 2.1 where $\alpha = m - 1$.

$$S_m(n) = n^{m+1} - m\left(\frac{1}{m}\sum_{i=1}^{m}\binom{m}{i-1}S_i(n-1)\right) = n^{m+1} - \sum_{i=1}^{m}\binom{m}{i-1}S_i(n-1)$$

Let $k = n - 1$

$$S_m(k+1) = (k+1)^{m+1} - \sum_{i=1}^{m}\binom{m}{i-1}S_i(k)$$

$$S_m(k+1) = (k+1)^{m+1} - \binom{m}{m-1}S_m(k) - \sum_{i=1}^{m-1}\binom{m}{i-1}S_i(k)$$

$$S_m(k) + (k+1)^m = (k+1)^{m+1} - mS_m(k) - \sum_{i=1}^{m-1}\binom{m}{i-1}S_i(k)$$

$$S_m(k) + mS_m(k) = (k+1)^{m+1} - (k+1)^m - \sum_{i=1}^{m-1}\binom{m}{i-1}S_i(k)$$

Hence,

$$S_m(k) = \frac{1}{m+1}\left(k(k+1)^m - \sum_{i=1}^{m-1}\binom{m}{i-1}S_i(k)\right)$$

The formula above directly implies a formula to find Bernoulli numbers. In this section, we shall derive such formula. The first thing we need to note is that B_{2n+1} always equals zero for all positive integer n. Another observation is that B_{2n} always appears as the coefficient of the first power of n in the polynomial $S_{2m}(n)$.

Theorem 3.2.: For B_{2k}, we have

$$B_{2k}=\frac{1}{2k+1}\left(\frac{1}{2}-\sum_{m=1}^{k-1}\binom{2k}{2m-1}B_{2m}\right)$$

Proof: From theorem 3.1, we have

$$S_m(n)=\frac{1}{m+1}\left(n(n+1)^m-\sum_{i=1}^{m-1}\binom{m}{i-1}S_i(n)\right)$$

Now we can expand the formula as follows

$$S_m(n)=\frac{1}{m+1}\left(n(n+1)^m-\frac{n(n+1)}{2}-\sum_{i=2}^{m-1}\binom{m}{i-1}S_i(n)\right)$$

Let $m = 2k$

$$S_{2k}(n)=\frac{1}{2k+1}\left(n(n+1)^{2k}-\frac{n(n+1)}{2}-\sum_{i=2}^{2k-1}\binom{2k}{i-1}S_i(n)\right)$$

Now, we separate the summation of the even terms and the odd terms

$$S_{2k}(n)=\frac{1}{2k+1}\left(n(n+1)^{2k}-\frac{n(n+1)}{2}-\sum_{m=1}^{k-1}\binom{2k}{2m-1}S_{2m}(n)-\sum_{m=2}^{k}\binom{2k}{(2m-1)-1}S_{2m-1}(n)\right)$$

Now, we only need to consider the coefficient of the first power of n. In the following, we now write B_{2k} instead of $S_{2k}(n)$ and we only write the coefficient of the first power of each polynomial we find. Therefore, the coefficient of n in the expansion $n(n + 1)^{2k}$ is 1 and the coefficient of n in $\frac{n(n+1)}{2}$ is $\frac{1}{2}$. Therefore, we write

$$B_{2k}=\frac{1}{2k+1}\left(1-\frac{1}{2}-\sum_{m=1}^{k-1}\binom{2k}{2m-1}B_{2m}-\sum_{m=2}^{k}\binom{2k}{(2m-1)-1}B_{2m-1}\right)$$

Since $B_{2m-1} = 0$ for all positive integers $m > 1$, the second summation term vanishes. Therefore,

$$B_{2k}=\frac{1}{2k+1}\left(\frac{1}{2}-\sum_{m=1}^{k-1}\binom{2k}{2m-1}B_{2m}\right)$$

3.2. Generalization of Theorem 3.1.

We shall generalize Theorem 3.1 to compute the sum of powers of an arithmetic progression. Firstly, we define $S_k^{a,b}(n):=\sum_{i=1}^{n}(ai+b)^k$. $S_1^{a,b}(n)$ can be easily computed by using the usual arithmetic series formula. In this subsection, we will show that we can compute $S_k^{a,b}(n)$ given $S_1^{a,b}(n)$, $S_2^{a,b}(n)$, ..., $S_{k-1}^{a,b}(n)$. However, we should firstly evaluate the more general form of the integral in Lemma 2.1 to prove the generalization. We will use the same approach as in the proof of Lemma 2.1.

Lemma 3.3.: For the following integral, we have

$$\int_1^n (at+b)^\alpha \lfloor t \rfloor dt=\frac{1}{\alpha+1}\sum_{i=1}^{\alpha+1}\binom{\alpha+1}{i-1}\left(S_i^{a,b}(n-1)-b\,S_{i-1}^{a,b}(n-1)\right)$$

Proof: Using the same approach as the proof of Lemma 2.1, we have

$$I=\int_k^{k+1}(at+b)^{\alpha}\lfloor t\rfloor dt=\sum_{k=1}^{n-1}k\int_k^{k+1}(at+b)^{\alpha}\,dt$$

$$I=\sum_{k=1}^{n-1}k\frac{(ak+b+a)^{\alpha+1}-(ak+b)^{\alpha+1}}{a(\alpha+1)}=\frac{1}{a(\alpha+1)}\sum_{k=1}^{n-1}\left(k\left((ak+b+a)^{\alpha+1}-(ak+b)^{\alpha+1}\right)\right)$$

Applying Binomial Expansion, we obtain

$$I=\frac{1}{\alpha+1}\sum_{k=1}^{n-1}\sum_{i=1}^{\alpha+1}\binom{\alpha+1}{i-1}ka^{\alpha-i+1}(ak+b)^{i-1}$$

$$=\frac{1}{\alpha+1}\sum_{i=1}^{\alpha+1}\binom{\alpha+1}{i-1}a^{\alpha-i+1}\sum_{k=1}^{n-1}k(ak+b)^{i-1}$$

$$=\frac{1}{\alpha+1}\sum_{i=1}^{\alpha+1}\binom{\alpha+1}{i-1}a^{\alpha-i}\sum_{k=1}^{n-1}(ak+b-b)(ak+b)^{i-1}$$

$$=\frac{1}{\alpha+1}\sum_{i=1}^{\alpha+1}\binom{\alpha+1}{i-1}a^{\alpha-i}\sum_{k=1}^{n-1}(ak+b)^{i}-b(ak+b)^{i-1}$$

Finally, we have

$$\int_1^n(at+b)^{\alpha}\lfloor t\rfloor dt=\frac{1}{\alpha+1}\sum_{i=1}^{\alpha+1}\binom{\alpha+1}{i-1}a^{\alpha-i}\left(S_i^{a,b}(n-1)-b\,S_{i-1}^{a,b}(n-1)\right)$$

Theorem 3.4.: Let $S_m^{a,b}(k)=\sum_{i=1}^{k}(ai+b)^m$, then

$$S_m^{a,b}(k)=\frac{1}{m+1}\left(k(ak+a+b)^m+bm\,S_{m-1}^{a,b}(k)-\sum_{i=1}^{m-1}a^{m-i}\binom{m}{i-1}\left(S_i^{a,b}(k)-b\,S_{i-1}^{a,b}(k)\right)\right)$$

Proof: Let $f(x)=(ax+b)^m$. By Abel Summation Formula,

$$\sum_{i=1}^{n}(ai+b)^m=n(an+b)^m-ma\int_1^n(at+b)^{m-1}\lfloor t\rfloor dt$$

Using Lemma 3.3, we rewrite the integral as follows

$$S_m^{a,b}(n)=n(an+b)^m-ma\left(\frac{1}{m}\sum_{i=1}^{m}a^{m-i-1}\binom{m}{i-1}\left(S_i^{a,b}(k)-b\,S_{i-1}^{a,b}(k)\right)\right)$$

Now, let

$n=k+1$

$$S_m^{a,b}(k+1)=(k+1)(ak+a+b)^m-a\sum_{i=1}^{m}a^{m-i-1}\binom{m}{i-1}\left(S_i^{a,b}(k)-b\,S_{i-1}^{a,b}(k)\right)$$

$$S_m^{a,b}(k)+(ak+a+b)^m=(k+1)(ak+a+b)^m-m\,S_m^{a,b}(k)+bm\,S_{m-1}^{a,b}(k)-\sum_{i=1}^{m-1}a^{m-i}\binom{m}{i-1}\left(S_i^{a,b}(k)-b\,S_{i-1}^{a,b}(k)\right)$$

$$(m+1)S_m^{a,b}(k)=k(ak+a+b)^m+bm\,S_{m-1}^{a,b}(k)-\sum_{i=1}^{m-1}a^{m-i}\binom{m}{i-1}\left(S_i^{a,b}(k)-b\,S_{i-1}^{a,b}(k)\right)$$

Hence, we get

$$S_m^{a,b}(k)=\frac{1}{m+1}\left(k(ak+a+b)^m+bm\,S_{m-1}^{a,b}(k)-\sum_{i=1}^{m-1}a^{m-i}\binom{m}{i-1}\left(S_i^{a,b}(k)-b\,S_{i-1}^{a,b}(k)\right)\right)$$

As a quick check, when $a = 1$ and $b = 0$, the formula reduces to the formula in Theorem 3.1.

3.3. Alternative Proof of Theorem 3.1.

Alternatively, Theorem 3.1 can be also derived from Pascal's Sum of Powers Identity (Pascal, 1964).

Theorem 3.5.: (Pascal's Sum of Powers Identity). Let $S_m(n)=\sum_{i=1}^{n}i^m$

$$(n+1)^{m+1}-1=\sum_{i=0}^{m}\binom{m+1}{i}S_i(n)$$

Alternative Proof of Theorem 3.1. For $m = k$, we have

$$(n+1)^{k+1}-1=(k+1)S_k(n)+\sum_{i=0}^{k-1}\binom{k+1}{i}S_i(n) \qquad \text{...(3)}$$

While for $m = k - 1$, we have

$$(n+1)^{k}-1=\sum_{i=0}^{k-1}\binom{k}{i}S_i(n) \qquad \text{...(4)}$$

Substracting Equation (4) from Equation (3),

$$(n+1)^{k+1}-(n+1)^k=(k+1)S_k(n)+\sum_{i=0}^{k-1}\left(\binom{k+1}{i}-\binom{k}{i}\right)S_i(n)$$

By Pascal's Binomial Coefficient Identity,

$$\binom{k+1}{i}-\binom{k}{i}=\binom{k}{i-1}$$

$$n(n+1)^k=(k+1)S_k(n)+\sum_{i=1}^{k-1}\binom{k}{i-1}S_i(n)$$

Rearranging,

$$S_k(n)=\frac{1}{k+1}\left(n(n+1)^k-\sum_{i=1}^{k-1}\binom{k}{i-1}S_i(n)\right)$$

4. Corollaries

Corollary 4.1.: For all odd numbers $2m$-1, we have $S_1(n)|S_{2m-1}(n)$

Proof: By Theorem 3.1,

$$S_{2m-1}(n)=\frac{1}{2m}\left(n(n+1)^{2m-1}-\sum_{i=1}^{2m-2}\binom{2m-1}{i-1}S_i(n)\right)$$

The above expression can be written as follows

$$S_{2m-1}(n)=\frac{n(n+1)^{2m-1}}{2m}-\sum_{i=1}^{2m-2}\binom{2m-1}{i-1}\frac{S_i(n)}{2m} \qquad \text{...(5)}$$

While for the sum of even power $2m$ we have

$$S_{2k}(n)=\frac{n(n+1)^{2k}}{2k+1}-\sum_{i=1}^{2k-1}\binom{2k}{i-1}\frac{S_i(n)}{2k+1} \qquad ...(6)$$

We see that if the above equation is multiplied by $\frac{1}{2m}$, it will be divisible by $S_1(n)=\frac{n(n+1)}{2}$. Therefore, if we substitute (6) to (5), it will be clear that $S_1(n)$ divides $S_{2m-1}(n)$

Remark: The fact that $S_1(n)|S_{2m-1}(n)$ is the underlying idea of Faulhaber's Formula for the sum of odd powers (Donald, 1993).

Corollary 4.2.: (Nicomachus' Theorem). The sum of first n cubes equals the square of the sum of the first n natural numbers

Proof: It is generally known that $S_1(n)=\frac{n(n+1)}{2}=\frac{n^2}{2}+\frac{n}{2}$. Now we shall apply Theorem 3.1 to find the formula for the sum of squares and the formula for the sum of cubes.

$$S_2(n)=\frac{1}{3}\left(n(n+1)^2-\sum_{i=1}^{1}\binom{2}{i-1}S_i(n)\right)$$

$$=\frac{n^3}{3}+\frac{2n^2}{3}+\frac{n}{3}-\frac{n^2}{6}-\frac{n}{6}$$

$$=\frac{n^3}{3}+\frac{n^2}{2}+\frac{n}{6}$$

Once again, we apply the formula to obtain the sum of cubes

$$S_3(n)=\frac{1}{4}\left(n(n+1)^3-\sum_{i=1}^{2}\binom{3}{i-1}S_i(n)\right)$$

$$=\frac{n(n+1)^3}{4}-\frac{n(n+1)}{8}-\frac{n^3}{4}-\frac{3n^2}{8}-\frac{n}{8}$$

Collecting the like terms and simplifying,

$$S_3(n)=\frac{n^4}{4}+\frac{n^3}{2}+\frac{n^2}{4}=\left(\frac{n(n+1)}{2}\right)^2$$

References

Beardon, A.F. (1996). Sums of Powers of Integers. *The American Mathematical Monthly*, 103(3), 201-213.

Tom M. Apostol. (2013). Introduction to Analytic Number Theory. *Springer Science and Business Media*.

Pascal, B. (1964). *Sommation des puissances numriques*. *Oeuvres Compltes*, 3, 341-367.

Donald E. Knuth. (1993). Johann Faulhaber and Sums of Powers. *Mathematics of Computation*, 61(203), 277-294.

Cite this article as: Ahmad Daffa I.F. Askari (2024). Recursive Formula for Sum of Powers of Natural Numbers and its Generalization to Arithmetic Progression. *International Journal of Pure and Applied Mathematics Research*, 4(2), 63-69. doi: 10.51483/IJPAMR.4.2.2024.63-69.