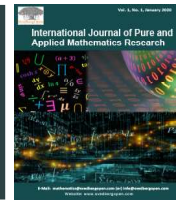




International Journal of Pure and Applied Mathematics Research

Publisher's Home Page: <https://www.svedbergopen.com/>



Research Paper

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A Generalisation of Pythagoras' Theorem

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Article Info

Volume 4, Issue 2, October 2024

Received : 11 June 2024

Accepted : 18 September 2024

Published : 05 October 2024

[doi: 10.51483/IJPAMR.4.2.2024.59-62](https://doi.org/10.51483/IJPAMR.4.2.2024.59-62)

Abstract

We provide a generalisation of Pythagoras' theorem, which enables us to calculate the absolute value of a hypersurface of arbitrary dimensions in terms of its projections. For a one dimensional hyperplane the theorem reduces to Pythagoras' theorem.

Keywords: *Pythagoras, Inner product, Hypersurface*

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1. A Generalisation Of Pythagoras' Theorem

It has been canonical to associate Pythagoras' theorem with the geometry of a triangle and De Gua's theorem and higher dimensional analogues, with the geometry of a simplex. We, however, adopt a more general geometric interpretation. Namely, we state a new theorem, which yields the absolute value of an hyperplane of arbitrary shape in terms of its components,

$$X^2 = \sum_{i_1 < \dots < i_p} x^{i_1 \dots i_p} x_{i_1 \dots i_p} \quad \dots(1)$$

where $x^{i_1 \dots i_p}$, X , p and d are the components of the hyperplane, the absolute value of the hyperplane, the dimension of the hyperplane and the overall dimension of the space respectively. We retrieve Pythagoras' theorem,

$$X^2 = x^2 + y^2$$

for $p = 1$ and $d = 2$. For $p = 2$ and $d = 3$, we obtain a new theorem, which gives the absolute value, X , of a plane of arbitrary shape, in terms of its components

$$X^2 = a^2 + b^2 + c^2,$$

where a , b and c are the projections of the plane, x^{12} , x^{13} and x^{23} , onto the three planes, spanned by the coordinate axis axes. This theorem has geometric meaning, as is illustrated (Bell, 1999).

2. Inner Product and Projections

The square of the absolute value of any hyperplane with arbitrary boundary is equal to sum of the squares of the

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absolute values of the projections of the hyperplane onto its basis elements. We argue that this insight is true in the trivial case, by realizing that we can always rotate any hyperplane, such that it has only one component.

For the sake of clarity, a hyperplane can be decomposed into its projections onto its basis elements as follows

$$x = \sum_{i_1 < \dots < i_p} x^{i_1 \dots i_p} e_{i_1} \wedge \dots \wedge e_{i_p} \tag{2}$$

where e_i are the basis vectors and where the components are the projections $x^{i_1 \dots i_p} = \langle e^{i_1} \wedge \dots \wedge e^{i_p}, x \rangle$. We recognize the theorem for hyperplanes (Bell, 1999) as an inner product of an hyper plane (Benson, 1999) with itself and derive the theorem this way. The inner product of the basis elements is equal to the Gram matrix

$$\langle e_1 \wedge \dots \wedge e_p, f_1 \wedge \dots \wedge f_p \rangle = \det \langle e_k, f_l \rangle$$

where $\langle e_i, e_j \rangle = g_{ij}$ and $e_i = g_{ij} e^j$. This determines the inner product of two p -forms, x , and y , in the following manner

$$\langle x, y \rangle = \sum_{i_1 < \dots < i_p} x^{i_1 \dots i_p} y^{j_1 \dots j_p} \langle e_{i_1}, f_{j_1} \rangle \dots \langle e_{i_p}, f_{j_p} \rangle \tag{3}$$

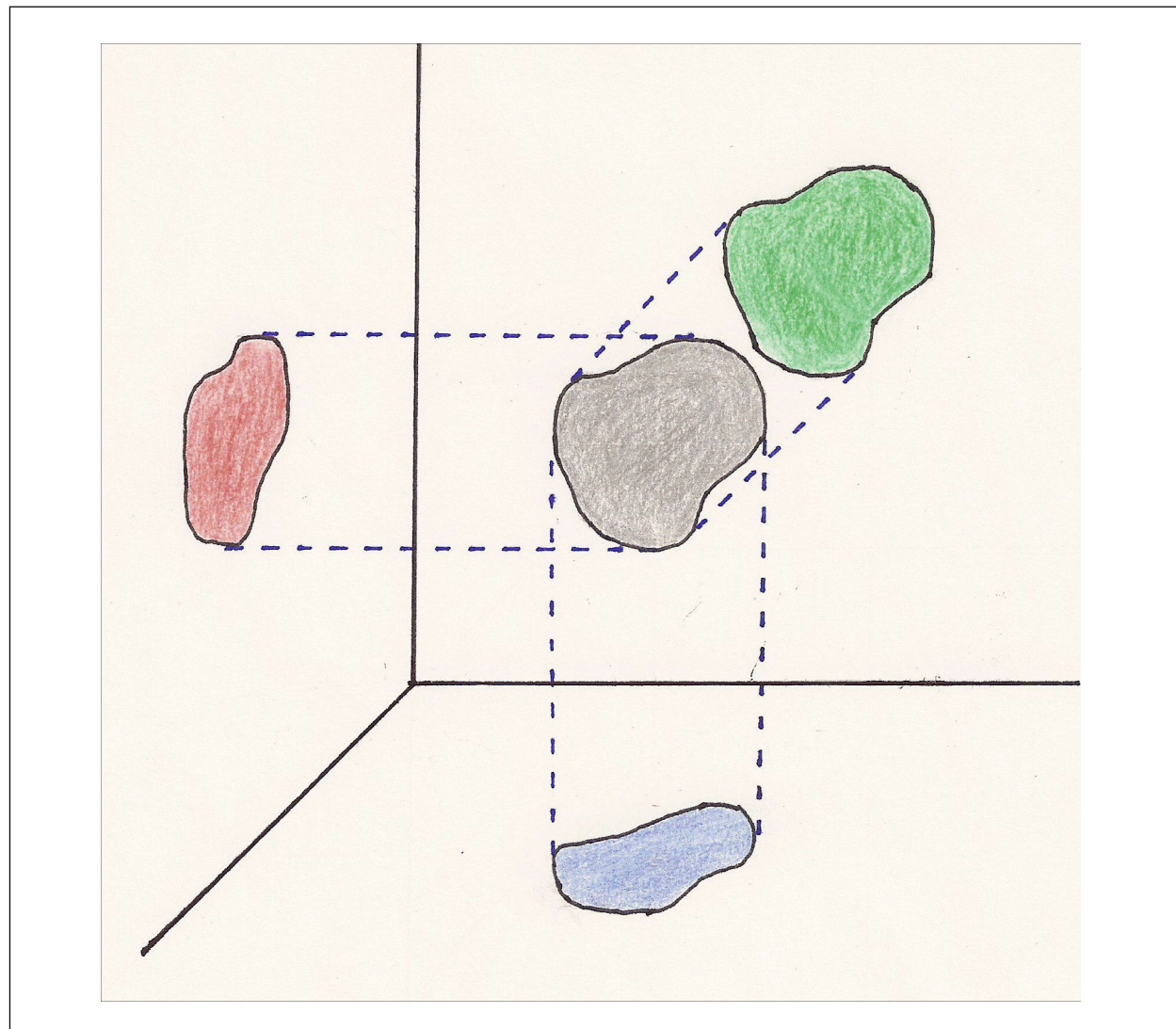


Figure 1: The Decomposition of a Grey Surface, into its Projections, x^{12} , x^{13} and x^{23} in Blue, Red and Green is Being Displayed

where the absence of a sum sign implies Einstein summation convention. We now derive the theorem for hyper planes (Bell, 1999),

$$\langle x, x \rangle = \sum_{i_1 < \dots < i_p} x^{i_1 \dots i_p} x_{i_1 \dots i_p} = \|x\|^2 = X^2$$

Note that the theorem holds in relativistic spaces.

3. Hypersurfaces

We generalize the theorem for hyperplanes to a theorem for hypersurfaces. Letting the components to be innitesimal, $x^{i_1 \dots i_p} = dx^{[i_1} \dots dx^{i_p]}$, yields

$$dX^2 = \sum_{i_1 < \dots < i_p} dx^{[i_1} \dots dx^{i_p]} dx^{[j_1} \dots dx^{j_p]} g_{i_1 j_1} \dots g_{i_p j_p} \tag{4}$$

where dX is the absolute value of an innitesimal hypersurface, where x^i are arbitrary coordinates and where the square brackets imply antisymmetrization. Whenever the metric is diagonal, we obtain,

$$dX^2 = \sum_{i_1 < \dots < i_p} \left(dx^{[i_1} \dots dx^{i_p]} \right)^2 g_{i_1 i_1} \dots g_{i_p i_p}$$

We conclude with three examples for clarication. We retrieve the line element,

$$dX^2 = g_{i,j} dx^i dx^j$$

for $p = 1$.

Taking the root of both sides of the theorem for hypersurfaces, we obtain the d -dimensional volume element,

$$dX = \sqrt{|g|} dx^{[1} \dots dx^{d]}$$

for $p = d$.

The calculation of the surface of a sphere goes as follows. Taking $p = 2$, $d = 3$ and choosing the coordinates to be spherical coordinates, gives us,

$$dX^2 = r^2 \sin^2 \phi (dr d\theta)^2 + r^2 (dr d\phi)^2 + r^4 \sin^2 \phi (d\theta d\phi)^2,$$

where the metric is given by $g_{rr} = 1$; $g_{\phi\phi} = r^2$ and $g_{\theta\theta} = r^2 \sin^2 \phi$. The surface of a sphere is characterized by the radius being constant, $dr = 0$. If we then take the root and put in the appropriate integration boundaries, we get the value for the area.

$$X = \int_0^{2\pi} \int_0^\pi r^2 \sin \phi (d\phi d\theta) = 4\pi r^2$$

4. Historical Discussion

We believe the theorem of Pythagoras to be a most important theorem and also its generalisation. One might question the novelty of this work though. The Namu-Goto action, for instance,

$$S = -\frac{T_0}{c} \int d\tau d\sigma \sqrt{(\dot{X} \cdot X')^2 - (\dot{X})^2 (X')^2}$$

also yields the absolute value of a two dimensional hypersurface, where $X^\mu = X^\mu(\tau, \sigma)$ is a vector, which determines the shape of the hypersurface. It gives the absolute value, but it is unaware of the projections, the righthand sight of our the theorem for hypersurfaces, a generalization of Pythagoras' theorem. This is true, because the vector X^μ is parametrized by σ and τ , and moves only on the surface.

There is actually an older physics equation, the electromagnetic energy, which only captures the righthand sight of the theorem for hypersurfaces. We can recognize $F^{\mu\nu}$ as the projection of an innitesimal surface on the x^μ, x^ν -plane. The absolute value of F is then given by $F^2 = F^{\mu\nu} F_{\mu\nu}$.

References

- Bell, John L. (1999). *The Art of the Intelligible: An Elementary Survey of Mathematics in its Conceptual Development*. Kluwer. ISBN 0-7923-5972-0
- Benson, Donald (1999). *The Moment of Proof: Mathematical Epiphanies, 172-173*, Oxford University Press.
- Euclid (1956). *The Thirteen Books of Euclid's Elements*, Translated from the Text of Heiberg, with Introduction and Commentary. 1 (Books I and II), Translated by Heath, Thomas L. (Reprint of 2nd (1925) Ed.). Dover. On-Line Text at archive.org
- Heath, Sir Thomas (1921). The 'Theorem of Pythagoras'. *A History of Greek Mathematics*, 2, 144, Dover Publications, Inc. (1981) (Ed.). Clarendon Press, Oxford. ISBN 0-486-24073-8.v
- James Overduin and Richard Conn Henry (2020). *Physics and the Pythagorean Theorem*, arXiv:2005.10671
- Libeskind, Shlomo (2008). *Euclidean and Transformational Geometry: A Deductive Inquiry*. Jones and Bartlett Learning. ISBN 978-0-7637-4366-6
- Loomis, Elisha Scott (1940). *The Pythagorean Proposition* (2nd Ed.), Ann Arbor, Edwards Brothers, Michigan. ISBN 9780873530361
- Maor, Eli (2007). *The Pythagorean Theorem: A 4,000-Year History*. Princeton, Princeton University Press, New Jersey. ISBN 978-0-691-12526-8
- Maor, Eli. (2019). *The Pythagorean Theorem: A 4,000-Year History*. 65, Princeton University Press.
- Neugebauer, Otto (1969). *The Exact Sciences in Antiquity* (2nd Ed.), Courier Dover Publications. ISBN 0-486-22332-9. OCLC 638685764
- Robson, Eleanor and Jacqueline Stedall (Eds.) (2009). *The Oxford Handbook of the History of Mathematics*, vii + 918, Oxford University Press, Oxford. ISBN 978-0-19-921312-2
- Stillwell, John (1989). *Mathematics and its History*. Springer-Verlag. ISBN 0-387-96981-0. Also ISBN 3-540-396981-0
- van der Waerden and Bartel Leendert (1983). *Geometry and Algebra in Ancient Civilizations*. Springer. ISBN 3-540-12159-5. Pythagorean Triples Babylonian Scribes van der Waerden.