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Introducing Novel Geometric Insights and Three-Dimensional Depictions of the Pythagorean Theorem for any Triangles

Marco Aurélio dos Santos Bernardes[1*]

[1]Department of Mathematics, Research Group Differential Equations and Dynamical Systems (DD), University of Hamburg, Bundesstraße 55, 20146, Hamburg, Germany. E-mail: marco.bernardes@uni-hamburg.de

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Abstract

The paper delves into a comprehensive exploration of geometric relationships within Euclidean geometry, focusing on the profound implications of connecting any point on a midsegment with the vertices of a generic triangle. Through meticulous analysis, it establishes that the internal triangles formed by these connections impeccably adhere to the principles of the Pythagorean theorem. This revelation not only reinforces fundamental geometric concepts but also highlights the elegant interplay between seemingly disparate elements within geometric constructs. Furthermore, the study extends its inquiry into the realm of three-dimensional geometry, unveiling Pythagorean relationships inherent within spatial triangles. This expansion into three-dimensional space offers a novel perspective, showcasing how geometric principles transcend traditional boundaries and manifest in complex spatial arrangements. The exploration of these three-dimensional constructs presents a compelling case for a generalized extension of the Pythagorean theorem, shedding light on a previously unexplored realm of geometric harmony. Central to this investigation is the identification of a unique spatial region characterized by harmonious area interrelations among triangles. This region serves as a focal point for understanding the intricate geometric relationships at play, offering valuable insights into the underlying structure of geometric space. By elucidating these spatial dynamics, the study not only advances our understanding of geometric theory but also opens new avenues for exploration within the field of Euclidean geometry.

Keywords: *Pythagorean theorem, Three-dimensional space, Triangles, Three-dimensional geometric shape, Triangle midsegment theorem*

1. Introduction

The Pythagorean theorem stands as a cornerstone of elementary mathematics, its significance echoing across centuries of mathematical inquiry and practical applications. Despite its seemingly simple statement, its ramifications are profound and far-reaching. The body of literature dedicated to the Pythagorean theorem is vast, spanning numerous proofs, historical investigations, and practical applications. However, the scope of this discussion merely scratches the surface

* *Corresponding author: Marco Aurélio dos Santos Bernardes, Department of Mathematics, Research Group Differential Equations and Dynamical Systems (DD), University of Hamburg, Bundesstraße 55, 20146, Hamburg, Germany. E-mail: marco.bernardes@uni-hamburg.de*

of this rich tapestry. To embark on our exploration, we turn our attention to a seminal work, referenced as (Loomis, 1968). Within its pages lies a comprehensive examination of the historical evolution of the Pythagorean theorem, a journey that traverses the annals of diverse cultures and civilizations. This scholarly endeavor illuminates not only the theorem's mathematical essence but also its cultural and historical significance, revealing how its understanding has evolved over time. One of the most compelling aspects of (Loomis, 1968) is its exhaustive compilation of over 370 proofs of the Pythagorean theorem. Each proof represents a unique perspective, employing distinct methodologies and mathematical techniques to unveil the truth behind this fundamental theorem. From elegant geometric constructions to sophisticated algebraic manipulations, these proofs offer a kaleidoscopic view of mathematical creativity and ingenuity. Beyond its mathematical elegance, the Pythagorean theorem finds resonance in various domains beyond the confines of pure mathematics. Loomis (1968) delves into the practical implications of the theorem, exploring its applications in geometry, physics, engineering, and beyond. Its versatility becomes apparent as it serves as a cornerstone for solving practical problems, ranging from architectural design to celestial navigation. Moreover, the book ventures into the realm of non-right triangles, transcending the traditional boundaries of the theorem. Through the lens of trigonometry and advanced geometric concepts, it elucidates how the Pythagorean framework can be extended to encompass triangles of all shapes and sizes, including acute and obtuse triangles. This extension broadens the theorem's utility, unlocking new vistas of mathematical exploration and problem-solving. In essence (Loomis, 1968) serves as a beacon illuminating the multifaceted nature of the Pythagorean theorem.

Kassie Smith in reference (2023) inquires about a more geometric proof for these concepts. Consequently, starting with the equation, they deduce that the combined area of two regular n-gons with side lengths a and b respectively, equates to the area of a regular n-gon with side length c. This can be shown by multiplying both sides of the equation by a specific constant, which is essentially the area of a regular n-gon with a side length of 1. Interestingly, for any given $n \geq 3$, this assertion also serves as a way to derive Pythagorean theorem in reverse. Within this context, the Wallace-Bolyai-Gerwien decomposition theorem (Wikipedia Contributors, 2023) becomes applicable. This theorem suggests the existence of a decomposition of the smaller n-gons into polygonal components, which can be rearranged to form the larger n-gon. However, it's important to note that the number of individual pieces involved in this process might be extensive. The reference (Navas, 2020) proposes still another proof which is nonstandard as he does not use neither squares nor similarity of triangles. The author introduces a geometric demonstration of the aforementioned assertion concerning equilateral triangles employing arithmetical operations of triangular shapes.

The interesting premise posited by (Teia, 2018) suggests that when x equals y, the particular instance of the three-dimensional iteration of the Pythagorean geometric gear can be assembled from multiple instances of its two-dimensional counterparts arranged along the three orthogonal planes. Consequently, it is demonstrated that akin to how two perpendicular lines delineate a hypotenuse, forming a right-angled triangle, aligning three triangles across orthogonal planes with a shared central point establishes a novel three-dimensional 'hypotenuse,' constituting a hexagon, and collectively shaping a truncated tetrahedron. Just as the hexagon emerges from three diagonal hypotenuses, the truncated tetrahedron arises from three isosceles right triangles. Thus, the truncated tetrahedron represents the three-dimensional counterpart of the two-dimensional isosceles right triangle. The Pythagorean gear amalgamates one-, two, and three-dimensional theorems, epitomizing a harmonious equilibrium between lengths, areas, and volumes. This equilibrium is delineated through orthogonal and diagonal orientations, forming distinct grids. This discourse holds practical implications, notably in the modern aerospace sector, particularly within computational fluid dynamics, where a grid composed of geometric elements, such as tetrahedra and prisms, termed 'cells,' facilitates the analysis of fluid behavior in proximity to or within a structure, such as an aircraft. Analogously, the Pythagorean gear embodies a similar concept, wherein a gear can represent an individual balance or contribute to a network of balances when interconnected truncated octahedrons fill space. The Pythagorean constant 2 operates as a fractal intermediary, facilitating the conversion of information—lengths, areas, and volumes—between orthogonal and diagonal grids.

The comprehensive analysis conducted by (Economou, 2001) delves deeply into the multifaceted nature of the direct product group C2C2C2. By meticulously examining its foundational principles within the realm of Pythagorean arithmetic, the study offers a succinct yet thorough overview of the group's intricate structure. Moreover, the exploration extends beyond theoretical frameworks to visually demonstrate the group's manifestation in three-dimensional space. Through compelling visual representations, the study elucidates the geometric properties inherent in the group, shedding light on its potential implications in various fields, including music notation. Of particular note is the examination's focus on four distinct decompositions of the group. Through meticulous enumeration and vivid illustration of all subsets, the study provides a comprehensive understanding of the group's compositional elements. Furthermore, the text doesn't merely stop at theoretical analysis but also suggests diverse practical applications of this structure. It encourages further systematic exploration in areas such as architectural design, where the insights gained from analyzing and synthesizing architectural forms using the principles of the C2C2C2 group could lead to innovative and aesthetically compelling designs. Overall, this study contributes significantly to our understanding of the C2C2C2 group, paving the way for both theoretical advancements and practical applications in various domains.

The author of the paper (Rajput, 2019) presents an intricate interplay between the 3-4-5 Pythagorean triangle and the 1:2 right-angled triangle, highlighting their perfect complementary nature. This classical geometric harmony not only validates the Golden Ratio but also uncovers a precise correlation between π and φ, rooted deeply in classical geometric principles. It showcased the distinctive characteristics of the $1:2:\sqrt{5}$ triangle, establishing it as the true embodiment of the Golden Ratio Triangle. Through meticulous examination, it became evident that the Golden Ratio permeates every aspect of this particular right triangle's geometry. This unique triangle, inherently imbued with the Golden Proportion, exhibits a close relationship with the regular pentagon. Not only does it offer a novel method for constructing a regular pentagon, but the pentagon itself reflects the proportions of $1:2:\sqrt{5}$ within its geometry. Moreover, just as the $1:2:\sqrt{5}$ triangle embodies the Golden Ratio, this research introduces a concept of specialized right triangles accurately representing each Metallic Mean. These Metallic Ratios find perfect representation within the generalized right triangle described herein. Additionally, this study unveils a hidden geometric connection: the precise complementary relationship between the $1:2:\sqrt{5}$ triangle and the 3-4-5 Pythagorean triple. Remarkably, these two right triangles not only emerge together through various geometric constructions but also complement each other perfectly. Through their interaction, they uniquely manifest the Golden Ratio. Combining equivalent-sized $1:2:\sqrt{5}$ triangles and 3-4-5 triples along their common hypotenuse reveals the exact value of the Golden Ratio. Connecting them along their shorter or longer legs also unveils the precise Pi:Phi Correlation, demonstrating the profound connection between these two triangles. Thus, the $1:2:\sqrt{5}$ Golden Triangle, both individually and in conjunction with the 3-4-5 Pythagorean triple, not only serves as the ultimate geometric proof of the Golden Ratio but also elucidates the Golden Link in Geometry—a $\pi:\varphi$ correlation rooted in classical geometric principles with unparalleled precision.

In his paper (Ayushmaan, 2021), Ayushmaan Gupta has meticulously explored several distinct methods for proving the Pythagorean Theorem. These varied approaches shed light on the depth and versatility of this fundamental geometric principle. One of the most renowned proofs discussed is the Windmill proof, also referred to as the bride's chair proof, famously attributed to Euclid, the esteemed Greek mathematician whose contributions have profoundly shaped the geometric landscape. This elegant proof method ingeniously demonstrates the theorem's validity through a series of rotations resembling the blades of a windmill, showcasing the timeless ingenuity of ancient mathematical thought. Another compelling approach examined in this paper involves drawing altitudes within the geometric configuration and leveraging the properties of similarity, commonly referred to as the proof by similar triangles. This method highlights the inherent geometric relationships that underpin the Pythagorean Theorem, emphasizing the significance of geometric congruence and proportionality in its proof. Furthermore, the research delves into Garfield's proof, attributed to James Garfield, the 20th President of the United States, who was not only a statesman but also a scholar with a keen interest in mathematics. Garfield's proof offers a unique perspective on the theorem, showcasing the interdisciplinary nature of mathematical inquiry and its intersection with historical figures and events. Additionally, the paper explores a modern approach to proving the Pythagorean Theorem utilizing differentials, a technique that was once considered challenging to demonstrate rigorously. By leveraging concepts from calculus and differential geometry, this method offers a novel perspective on the theorem's underlying principles, illustrating the evolving nature of mathematical reasoning and its applications across diverse domains. Through the examination of these varied proof methods, this research underscores the richness and complexity inherent in the Pythagorean Theorem, showcasing its enduring relevance and profound implications across mathematical disciplines. Each proof method offers a unique lens through which to appreciate the elegance and universality of this fundamental geometric principle, highlighting the dynamic interplay between historical insights, contemporary scholarship, and mathematical innovation.

In the ensuing discourse, we aim to elucidate two fundamental theorems pertaining to the utilization of the Geometric Pythagorean relationship. The initial theorem delves into the examination of triangles fashioned by linking a point within the midsegment to the vertices of the triangle. This theorem scrutinizes the geometric configurations arising from such constructions and delineates their inherent properties and characteristics.

The second theorem ventures into the realm of non-coplanar triangles, where connections are established between the sides of the original triangle and a point situated in the spatial domain. This theorem explores the intricate interplay between geometric elements in three-dimensional space, elucidating the geometric implications and ramifications of these non-coplanar configurations. Through rigorous analysis and mathematical reasoning, we seek to unveil the underlying principles governing these geometric arrangements and their relevance in geometric theory and practice.

2. Pythagorean Concept for Coplanar Triangles Formed from a Point in the Midsegment and the Vertices of a Triangle

Theorem 1: (Geometric Pythagorean Relationships within Triangular Midsegment Compositions). In Euclidean geometry, for any triangle ΔABC and a point M lying on the midsegment connecting the midpoints of sides AC and BC, as depicted in Figure 1, the relationship between the areas of the three triangles ΔAMC, ΔAMB, and ΔBMC, is governed by the Geometric Proof of the Pythagorean theorem. Specifically, the sum of the areas of the triangles ΔAMC and ΔBMC is equal to the area of the triangle ΔAMB.

2.1. Definitions and Proof

Consider a generic triangle ΔABC, where AB is the base, and DE is the midsegment connecting the midpoints of sides AC and BC, as illustrated in Figure 1.

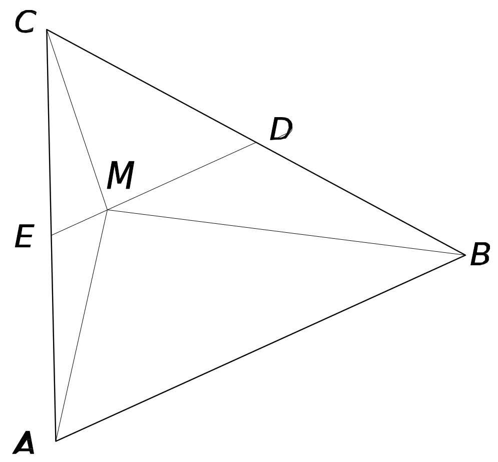

Figure 1: Generic Triangle ΔABC

The Triangle Midsegment Theorem elucidates the properties of triangles and the relationship between their midpoints. According to this theorem, when the midpoints of the sides of a triangle are connected, four identical smaller triangles are formed.

Let M be any point lying on the midsegment DE. If the area of each triangle, formed by connecting the midpoints of the sides of a given triangle, is one-fourth the area of the original triangle ΔABC, the following statements for the internally constructed triangles further align with the Triangle Midsegment Theorem and its underlying principles. Thus, it holds true that

$$ar(\Delta CDE) = \frac{1}{4} ar(\Delta ABC) \quad ...(1)$$

$$ar(\Delta CDE) = ar(\Delta CME) + ar(\Delta CMD) \quad ...(2)$$

$$ar(\Delta AMC) = 2ar(\Delta CME) = 2ar(\Delta AME) \quad ...(3)$$

$$ar(\Delta BMC) = 2ar(\Delta BMD) = 2ar(\Delta CMD) \quad ...(4)$$

The summation of the areas of the triangles ΔAMC and ΔBMC is given by Equation 5

$$ar(\Delta AMC) + ar(\Delta BMC) = 2ar(\Delta CME) + 2ar(\Delta CMD)$$

$$= \frac{1}{4} ar(\Delta ABC) + \frac{1}{4} ar(\Delta ABC) = \frac{1}{2} ar(\Delta ABC) \quad ...(5)$$

Since

$$ar(\Delta\, AMB) = \frac{1}{2} ar(\Delta\, ABC) \quad ...(6)$$

it can be demonstrated that, regardless of the size or shape of a triangle and for any point situated on the midsegment, the relationship between the areas of the three resultant internal triangles follows the principles of the Geometric Proof of the Pythagorean theorem, i.e., the sum of the areas of two smaller internal triangles given by Equation 5 is always equivalent to the area of the larger internal triangle provided by Equation 6. Hence,

$$ar(\Delta AMC) + ar(\Delta BMC) = ar(\Delta AMB) \quad ...(7)$$

2.2. Special Case for M = D

For the case shown in Figure 2 where $M = D$, it holds true that

$$ar(\Delta BMC) = 0 \quad ...(8)$$

Unsurprisingly, the relationship between the internal triangle continues to hold Equation 9 assertion.

$$ar(\Delta AMB) = ar(\Delta AMC) \quad ...(9)$$

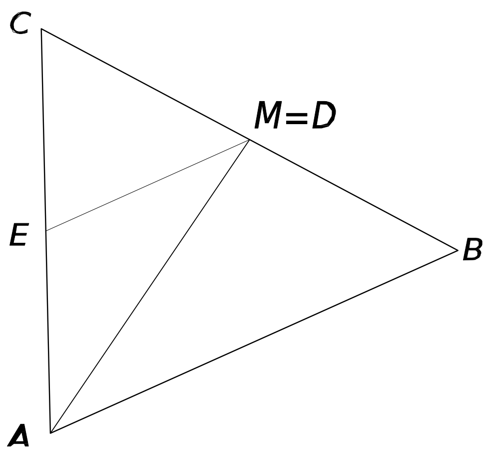

Figure 2: Special Case for *M* = *D*

2.3. Theorem Special Case for M ∉ ΔABC

In the situation depicted in Figure 3, where M is not inside the ΔABC, the Pythagorean structure undergoes a transformation such that the Equation 7 becomes the relationship shown in Equation 10.

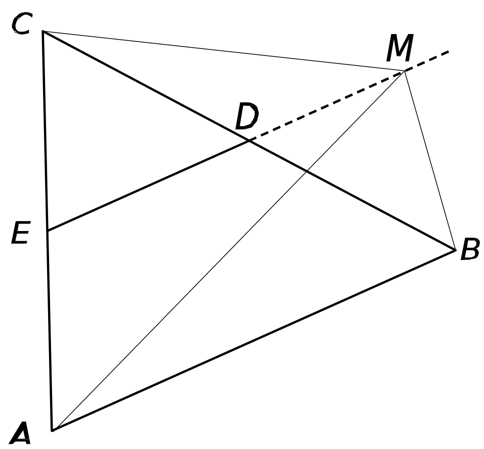

Figure 3: Theorem Special Case for *M* ∉ Δ*ABC*

$$ar(\Delta AMC) - ar(\Delta BMC) = ar(\Delta AMB) \quad ...(10)$$

Precisely, the triangle ΔAMC has been enlarged to become the largest among the three triangles, ensuring the preservation of the Pythagorean composition.

$$ar(\Delta AMB) + ar(\Delta BMC) = ar(\Delta AMC) \quad ...(11)$$

3. Extrapolating the Pythagorean Structure into the 3D Space

Theorem 2: (Pythagorean Relationships in Three-Dimensional Geometric Structures Formed by Three-Dimensional Spatial Triangular Compositions). In a specific curvature-domain shape within three-dimensional space, a geometric configuration is established wherein the summation of the areas of two constituent triangles created by connecting the vertices of a generic triangle ΔABC with a point $P(x, y, z)$ in 3D space, specifically triangle ΔAPB and triangle ΔBPC, is invariably equivalent to the area of the encompassing triangle ΔAPC for any selection of points A, B, and C that form said triangles, equivalent to the geometric demonstration of the Pythagorean Theorem.

3.1. Definitions and Proof

Now let's examine the general triangle ΔABC depicted in Figure 4, in which AB, AC, and BC represent its sides. The ΔAPB, ΔBPC and ΔAPC arise by connecting the generic point $P(x, y, z)$ in three-dimensional space with AB, AC, and BC.

The altitude h and the altitude distance l to the axis y of the triangle ΔABC are here determined by the its area, computed using the compact Heron's formula.

$$ar(\Delta ABC) = \frac{1}{4}\sqrt{4a^2b^2 - \left(a^2 + b^2 - c^2\right)^2} \quad ...(12)$$

$$h = \frac{2ar(\Delta ABC)}{a} \quad ...(13)$$

$$l = \sqrt{b^2 - h^2} \quad ...(14)$$

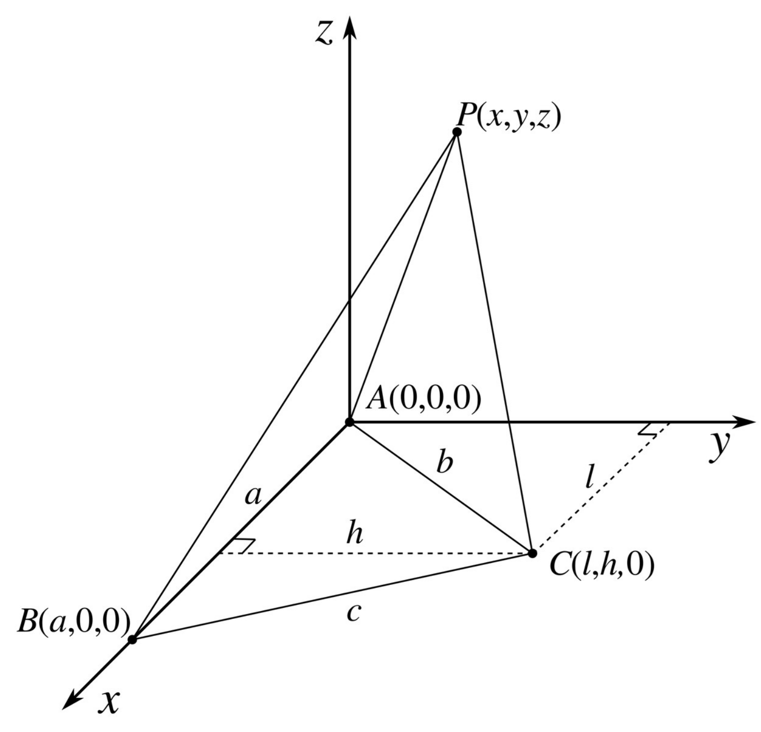

Figure 4: 3D Pythagorean Composition from ΔABC

Hence, h and l may be expressed in the following manner:

$$h=\frac{\sqrt{4a^2b^2-\left(a^2+b^2-c^2\right)^2}}{2a} \quad ...(15)$$

$$l=\frac{c^2-b^2-a^2}{2a} \quad ...(16)$$

We express the demonstration through a computationally efficient equation that necessitates only a single square root operation. The area of a triangle in coordinate geometry can calculated by the Equation 17, where (x_1, y_1, z_1), (x_2, y_2, z_2), and (x_3, y_3, z_3) are the vertices of a triangle in three-dimensional space.

$$ar(\Delta)=\frac{1}{2}\sqrt{\begin{array}{l}\left((x_2y_1)-(x_3y_1)-(x_1y_2)+(x_3y_2)+(x_1y_3)-(x_2y_3)\right)^2\\+\left((x_2z_1)-(x_3z_1)-(x_1z_2)+(x_3z_2)+(x_1z_3)-(x_2z_3)\right)^2\\+\left((y_2z_1)-(y_3z_1)-(y_1z_2)+(y_3z_2)+(y_1z_3)-(y_2z_3)\right)^2\end{array}} \quad ...(17)$$

In the subsequent iterations, we use Equation 17 to calculate the area of the triangles ΔAPB, ΔBPC and ΔAPC respectively. For the triangle ΔAPB where $x_1 = 0, y_1 = 0, z_1 = 0, x_2 = a, y_2 = 0, z_2 = 0, x_3 = x, y_3 = y, z_3 = z.$

$$ar(\Delta\, APB)=\frac{a}{2}\sqrt{y^2+z^2} \quad ...(18)$$

Regarding the triangle ΔBPC where $x_1 = l, y_1 = h, z_1 = 0, x_2 = a, y_2 = 0, z_2 = 0, x_3 = x , y_3 = y, z_3 = z$

$$ar(\Delta\, BPC)=\frac{\sqrt{\beta z^2+\left(2a^2y+\alpha y-a\sqrt{\beta}+\sqrt{\beta}x\right)^2+\left(2a^2z+\alpha z\right)^2}}{4a} \quad ...(19)$$

with

$$\alpha = a^2 + b^2 - c^2 \quad ...(20)$$

$$\beta = 4a^2b^2 - \alpha^2 \quad ...(21)$$

Concerning the triangle ΔAPC where $x_1 = l, y_1 = h, z_1 = 0, x_2 = 0, y_2 = 0, z_2 = 0, x_3 = x, y_3 = y, z_3 = z.$

$$ar(\Delta\, APC)=\frac{\sqrt{\alpha^2z^2+\beta z^2+\left(\alpha y+\sqrt{\beta}x\right)^2}}{4a} \quad ...(22)$$

Utilizing the Pythagorean geometric proof concept within the context of the irregular tetrahedron $ABCP$ yields three distinct solutions, as elucidated by Equation 23.

$$ar(\Delta APB) = ar(\Delta BPC) + ar(\Delta APC)$$

$$ar(\Delta BPC) = ar(\Delta APB) + ar(\Delta APC)$$

$$ar(\Delta APC) = ar(\Delta APB) + ar(\Delta BPC) \quad ...(23)$$

By substituting the terms from Equations 18, 19 and 22 into 23, we derive three respective implicit solutions for the hypothesized theorem tailored specifically to tetrahedrons named here as

3.2. Pythagorean Tetrahedron

$$2a^2\sqrt{y^2+z^2}-\sqrt{\beta z^2+\left(2a^2y+\alpha y-a\sqrt{\beta}+\sqrt{\beta}x\right)^2+\left(2a^2z+\alpha z\right)^2}$$
$$-\sqrt{\alpha^2z^2+\beta z^2+\left(\alpha y+\sqrt{\beta}x\right)^2}=0 \quad ...(24)$$

$$-2a^2\sqrt{y^2+z^2}+\sqrt{\beta z^2+\left(2a^2y+\alpha y-a\sqrt{\beta}+\sqrt{\beta}x\right)^2+\left(2a^2z+\alpha z\right)^2}$$

$$-\sqrt{\alpha^2z^2+\beta z^2+\left(\alpha y+\sqrt{\beta}x\right)^2}=0 \quad ...(25)$$

$$-2a^2\sqrt{y^2+z^2}-\sqrt{\beta z^2+\left(2a^2y+\alpha y-a\sqrt{\beta}+\sqrt{\beta}x\right)^2+\left(2a^2z+\alpha z\right)^2}$$

$$+\sqrt{\alpha^2z^2+\beta z^2+\left(\alpha y+\sqrt{\beta}x\right)^2}=0 \quad ...(26)$$

It is important to note that the concepts introduced here should not be confused with the concept of the Trirectangular Tetrahedron or with De Gua's theorem (de Gua de Malves, 1740). De Gua's theorem asserts that when a tetrahedron contains a right-angle corner (similar to a corner on a cube), the sum of the squares of the areas of the three faces adjacent to the right-angle corner is equal to the square of the area of the face positioned opposite to that corner.

4. Graphic Representation of the Resultant Curvature

In this section, we offer a visual exploration of the solution space provided by the implicit equations labeled as Equations 24, 25, and 26, which directly relate to the conjectured theorem discussed in section. Extrapolating the Pythagorean structure into the 3D Space. Leveraging the computational capabilities of a Computer Algebra System (CAS), we present three-dimensional graphical representations that vividly depict the solutions to these equations. Through these visualizations, we aim to provide intuitive insights into the geometric relationships and patterns embedded within the theorem under consideration.

We commence the visual exploration of resultant curvature by utilizing the classic 3-4-5 right triangle as our foundation. Through graphical representation of equations Equations 24, 25, and 26, we unveil the emergence of three distinct bell-shaped tri-dimensional curvatures extending outward from the midsegments of triangle ΔABC. This depiction is vividly illustrated in Figure 5. Foreseen is the indefinite expansion of these bell-curvatures, suggesting their potential for perpetual growth. The void spaces within this geometric configuration encompass not only the internal triangle enclosed by the midsegments but also three supplementary regions generated by extending these midsegments in a manner that circumvents intersection with the vertices of the prototypical triangle.

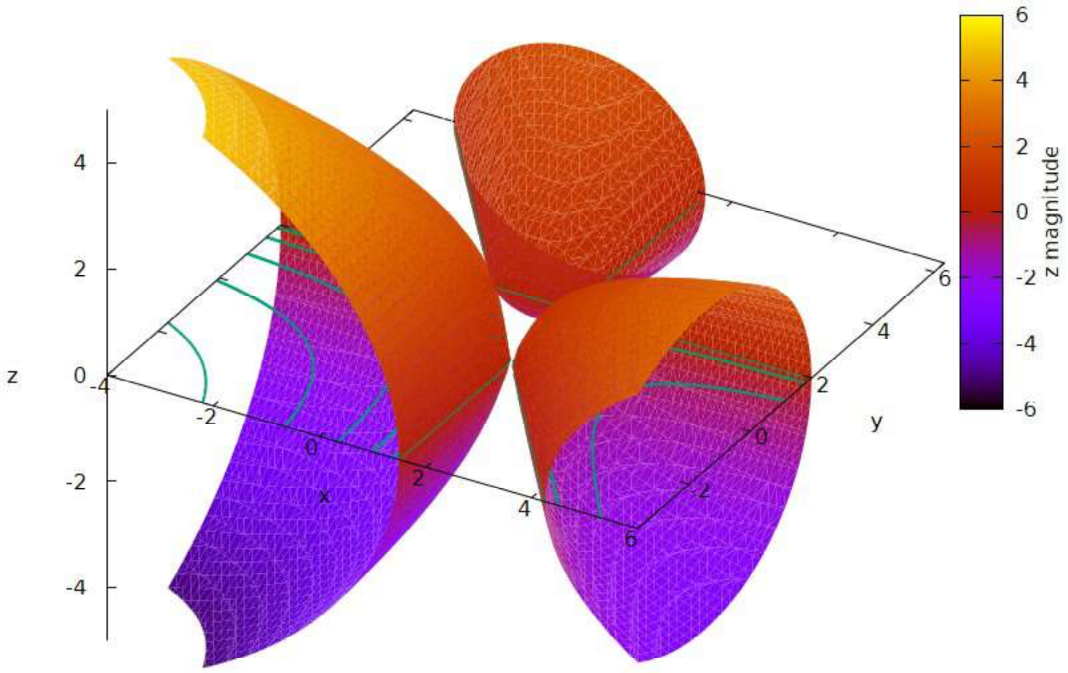

Figure 5: Visual Representation of the Resulting Curvature and Contour Levels for a 3-4-5 Right Triangle

Figures 6, 7 and 8 provide visual representations of the projection of the resultant surface onto the *xy*-plane. In these figures, the original triangle is highlighted using bold black lines, while the contour levels are delineated by green lines for 3-4-5 right triangles, 3-3-3 isosceles triangles, and 3-4-6 scalene triangles. The conjectured Theorem 1, elucidated in Section "Pythagorean concept for coplanar triangles formed from a point in the midsegment and the vertices of a triangle", can be easily substantiated by examining the midsegments of these triangles.

Expanding further, the projections offer a comprehensive view of how the resulting surface interacts with the *xy*-plane, showcasing the intricate relationships between the original triangle and its transformed counterparts. The use of

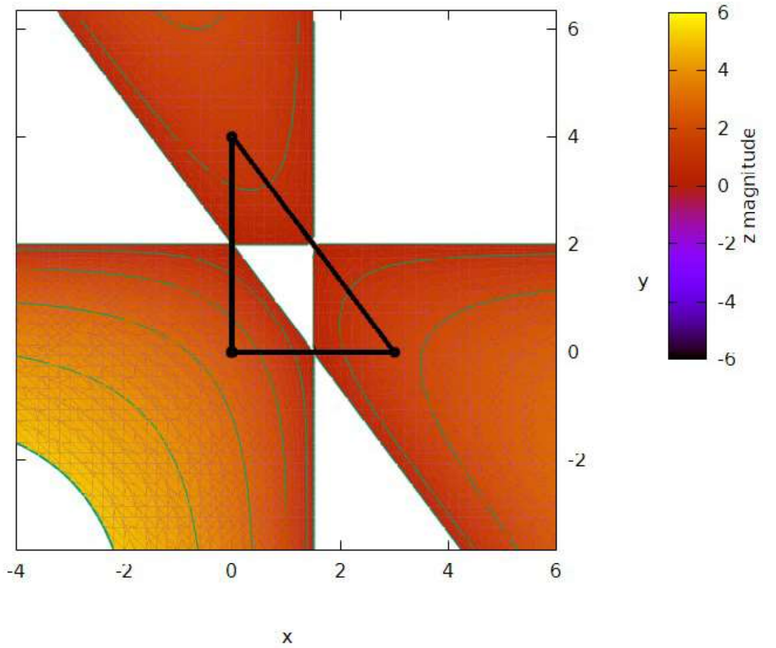

Figure 6: The 3-4-5 Right Triangle and the *xy*-Plane View of its Resulting Curvature and Contour Levels

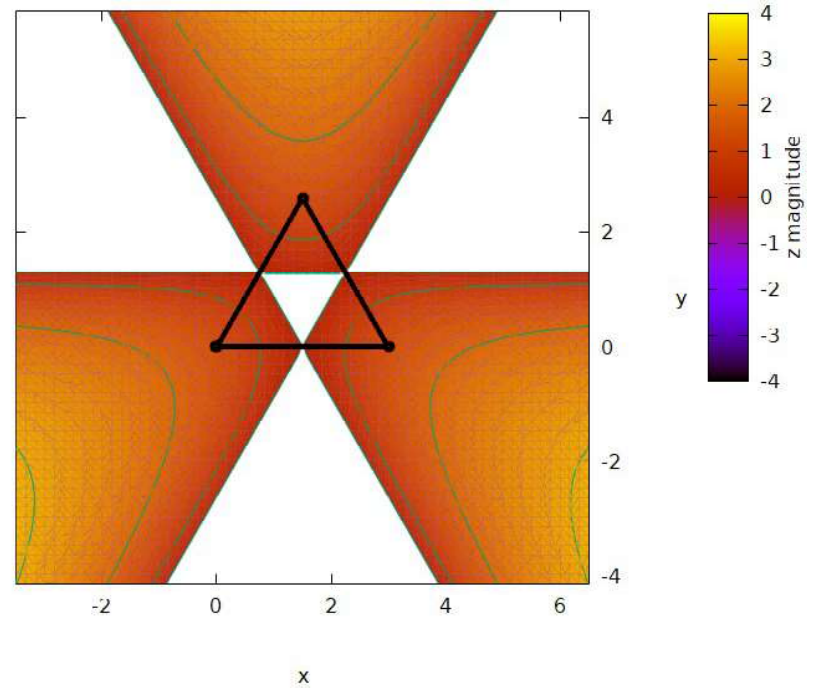

Figure 7: A 3-3-3 Isosceles Triangle and the *xy*-Plane View of its Resulting Curvature and Contour Levels

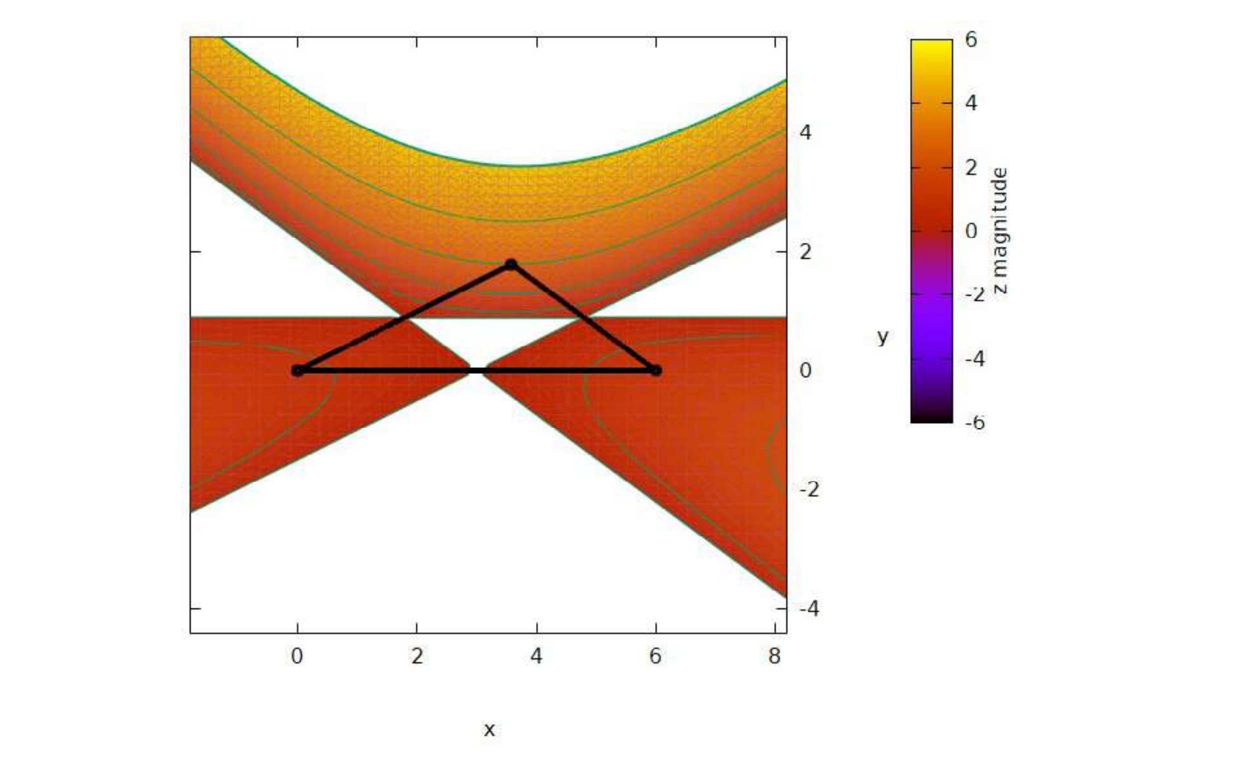

Figure 8: A 3-4-6 Scalene Triangle and the *xy*-Plane View of its Resulting Curvature and Contour Levels

bold black lines emphasizes the foundational structure of the original triangle, serving as a reference point for comparison. Meanwhile, the varying contour levels, depicted by green lines, highlight the distinct characteristics of each triangle type, facilitating a nuanced analysis of their geometric properties.

By scrutinizing the midsegments within these projections, one can discern patterns and relationships that affirm the conjectured Theorem 1. The alignment and distribution of midsegments provide compelling evidence of the underlying geometrical principles at play, further reinforcing the validity of the theorem as proposed in Section "Pythagorean concept for coplanar triangles formed from a point in the midsegment and the vertices of a triangle".

Overall, the visualizations presented in Figures 6, 7 and 8 offer valuable insights into the geometric transformations and relationships explored within this study.

Thus, as posited, our demonstration has elucidated that the trio of inner triangles formed by joining a point on a midsegment with the vertices of the triangle adheres rigorously to the geometric validation of the Pythagorean theorem. This discovery not only reinforces the timeless relevance of the Pythagorean theorem but also sheds light on a deeper geometric connection. This connection, reminiscent of an expanded interpretation of the Pythagorean theorem, unveils a distinct spatial configuration defined by the intricate interplay of triangle areas within any given triangle.

Moving forward, it becomes imperative for future endeavors to focus on the refinement and development of explicit formulas corresponding to the newly introduced concepts within our current framework. By pursuing this avenue of research, we can delve deeper into the mathematical intricacies and unveil further insights into the fundamental relationships that govern geometric structures. Such endeavors promise to enrich our understanding of geometry and pave the way for new avenues of exploration within the realm of mathematical theory and application.

5. Applications

The theorem introduced here, also referred to as hyper plane arrangements, and holds numerous applications across various fields:

Geometry: It provides a fundamental tool for understanding the intersection of curves and surfaces in geometric contexts. This is particularly relevant in algebraic geometry and differential geometry, where the theorem aids in studying the behavior of algebraic varieties and their intersections.

Computer Graphics: The theorem can be utilized in computer graphics for tasks such as collision detection and rendering. By understanding the intersections of various geometric primitives like lines, curves, and surfaces, computer graphics algorithms can efficiently determine when objects intersect or overlap in virtual environments.

Robotics and Computer Vision: In robotics and computer vision, this theorem can be applied to analyse the configuration spaces of robotic arms and the intersections of visual rays in imaging systems. This knowledge is crucial for tasks like path planning for robots and camera calibration for computer vision systems.

Optimization: The theorem can also be employed in optimization problems involving the arrangement of geometric objects. By understanding the intersections of these objects, optimization algorithms can efficiently find solutions to various problems in areas such as operations research, logistics, and engineering design.

Physics and Engineering: The theorem can find applications in physics and engineering, particularly in fields dealing with the modelling and analysis of physical systems. It is used to analyze the intersection of trajectories, surfaces, and other geometric entities in systems ranging from mechanical structures to electromagnetic fields.

Overall, the theorem has broad applicability in diverse fields, serving as a foundational concept for analyzing intersections in geometric arrangements and providing valuable insights into complex systems and structures.

6. Conclusion

In conclusion, this paper provides a significant contribution to the field of Euclidean geometry by revealing profound geometric relationships and extending the principles of the Pythagorean theorem into new and intriguing territories. Through a rigorous exploration of both two-dimensional and three-dimensional constructs, we have demonstrated that the connections between points on a midsegment and the vertices of a triangle yield internal structures that align perfectly with the Pythagorean theorem.

This discovery reinforces the interconnectedness of geometric concepts and underscores the inherent beauty within geometric constructs. Moreover, the extension of these relationships into three-dimensional space not only validates the timeless relevance of the Pythagorean theorem but also propels our understanding of spatial geometry to a higher level. The identification of a unique spatial region marked by harmonious area interrelations among triangles serves as a powerful tool for examining and interpreting complex geometric relationships.

By advancing these findings, our study paves the way for future research in Euclidean geometry and beyond. The insights gained from this exploration can inspire further investigations into generalized geometric theorems, providing a deeper comprehension of the structural fabric of geometric space. The presented applications in the fields of Geometry, Computer Graphics, Robotics and Computer Vision, Optimization, Physics, and Engineering are quite promising, showing the potential for these geometric principles to impact a broad range of scientific and engineering domains.

Ultimately, this work exemplifies the elegance and coherence of geometric theory, reinforcing its role as a fundamental pillar in the broader landscape of mathematics.

Acknowledgment

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