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Global Generalized Mersenne Numbers: Characterization and Distribution of Composites

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Abstract

In a previous paper, a new generalized definition of Mersenne numbers was proposed of the form $(a^n - (a - 1)^n)$ called Global Generalized Mersenne numbers, or Generalized Mersenne numbers in short. For prime exponents n, Generalized Mersenne primes and composites are generated. In this paper, the properties and distributions of Generalized Mersenne composites are investigated. It is found that the distribution of composite Generalized Mersenne numbers follow simple laws demonstrated in three theorems, as composite $GM_{a,n}$ appear periodically in an infinite number of groups of pairs of solutions in a, embedded into each others. It is remarkable that the distribution of composite $GM_{a,n}$ is completely characterized once the first values of a yielding composite $GM_{a,n}$ are found, as composite $GM_{a,n}$ are spaced regularly, separated by intervals of values depending on their factors $c_1 = 2nf_1 + 1$. Three methods are presented to calculate composite $GM_{a,n}$ and applied for the first six prime exponents n from 3 to 17.

Keywords: *Mersenne numbers, Generalized mersenne numbers, Distribution of generalized mersenne composites*

1. Introduction

A Mersenne number M_n is number of the form $M_n = (2^n - 1)$. For n is prime, some M_n are primes and only for n prime. However, the reciprocal does not hold as for some n primes, Mersenne numbers are not primes, like, e.g., for $n = 11$, M_{11} is composite, $M_{11} = 2047 = 23 \cdot 89$ (for review, see e.g., Ribenboim, 1989, Caldwell, 2021, Weisstein, 2023).

Several generalizations of Mersenne numbers have been proposed under various forms, by Crandall (1992), Solinas (1999, 2005, 2011), De Jesus Angel and Morales-Luna (2006), Hoque and Saikia (2014, 2015), Deng (2004). In a first paper (Pletser, 2024a), we proposed a new generalized definition of Mersenne numbers of the form $(a^n - (a - 1)^n)$ where the base a and the exponent n are natural integers. This new generalization is unrelated to previous ones. Although the

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name Generalized Mersenne number is already in use for pseudo Mersenne numbers used in cryptography (see a review in (Pletser, 2024a)), we propose to call them Global Generalized Mersenne numbers, or in short Generalized Mersenne numbers or $GM_{a,n}$, indexed by the base a and the exponent n that can take any natural integer values larger than 1. Generalized Mersenne numbers $GM_{a,n}$ are defined as the difference of the n^{th} power of two successive natural integers.

$$GM_{a,n} = a^n - (a-1)^n \qquad \text{...(1.1)}$$

for all $a \geq 2$ and $n \geq 2$ natural integers. As for Mersenne numbers, $GM_{a,n}$ generated with composite exponents n yield only composite $GM_{a,n}$, while prime exponents n yield composite and prime $GM_{a,n}$. Similarly to Mersenne numbers that can be written in the form

$$M_n = 2nq + 1 \qquad \text{...(1.2)}$$

with q natural integer, all $GM_{a,n}$ can also be written as

$$GM_{a,n} = 2nQ_n(a) + 1 \qquad \text{...(1.3)}$$

for all prime exponents $n \geq 3$ and for all natural integer values of the base $a \geq 2$, and where $Q_n(a)$ is a polynomial in a of degree $n - 1$

$$Q_n(a) = \Delta\left(\sum_{i=0}^{\frac{n-3}{2}}\left[(2\Delta)^i S_{n-2(i+1)}^{(i+1)}\right]\right) \qquad \text{...(1.4)}$$

where Δ is written for the triangular number of $(a-1)$, $\Delta(a-1) = \dfrac{a(a-1)}{2}$, and where natural integer coefficients $S_{n-2(i+1)}^{(i+1)} = \dfrac{(n-(i+2))!}{(i+1)!(n-2(i+1))!}$ depend only on n. $GM_{a,n}$ can be written more generally as

$$GM_{a,n} = 2n\left(\Delta Q_n'(2\Delta)\right) + 1 \qquad \text{...(1.5)}$$

for all prime exponents $n \geq 3$ and as

$$GM_{a,n} = 2n\left(\Delta(2\Delta+1)Q_n''(2\Delta)\right) + 1 \qquad \text{...(1.6)}$$

for all prime exponents $n \geq 5$ and for all natural integer values of the base $a \geq 2$, where $Q_n'(2\Delta)$ and $Q_n''(2\Delta)$ are polynomials in the variable $\Delta(a-1)$ only and of degrees $\left(\dfrac{n-3}{2}\right)$ and $\left(\dfrac{n-5}{2}\right)$ respectively, derived in (Pletser, 2024a). Several theorems on Mersenne numbers were generalized and new theorems were demonstrated for $GM_{a,n}$ related to congruence properties of $GM_{a,n}$ and of their factors (Pletser, 2024a).

In this paper, we investigate properties of $GM_{a,n}$ for n prime, $n \geq 3$, as we want to find where the prime $GM_{a,n}$ are. But instead of searching directly for the primes, we ask first what characterize the composite $GM_{a,n}$. Several methods and theorems are proposed in Section 2 to search for composite $GM_{a,n}$ and their properties and distributions are characterized. These methods are applied in Section 3 to the cases of the first six odd prime exponents n from 3 to 17. Conclusions are drawn in the last section.

Generalized Mersenne primes and their distributions are treated in a third companion paper.

2. Materials and Methods: General Search Methods for Generalized Mersenne Composites

Several methods are presented generally to determine whether a $GM_{a,n}$ is prime or composite. The examples given further for the different cases of n prime will show the simplicity of these methods. The composite $GM_{a,n}$ can be written generally as products of their natural integer factors c_i as:

$$GM_{a,n} = c_1^{e_1} c_2^{e_2} \dots c_i^{e_i} \dots \qquad ...(2.1)$$

where e_i are positive natural integer exponents and c_i are always of the form (Pletser, 2024a)

$$c_i = 2nf_i + 1 \qquad ...(2.2)$$

with f_i natural integers. For the general case of two factors and with $e_1 = e_2 = 1$ (i.e., even in the case of more than two factors and for e_1 and $e_2 \neq 1$, any combination of products of factors c_i can be represented in all generality by this case), one has

$$GM_{a,n} = (2nf_1 + 1)(2nf_2 + 1) = 2nQ_n(a) + 1 \qquad ...(2.3)$$

where

$$Q_n(a) = f_1 + 2nf_1f_2 + f_2 \qquad ...(2.4)$$

Note that, if f_1 or f_2 is nil and if the factors c_i cannot be further decomposed in other smaller factors of the form (2.2), the $GM_{a,n}$ is obviously prime. For prime exponents n, the bases a are searched for which the $GM_{a,n}$ are composites, i.e., such as relation (2.4) holds, i.e., to find a pair of natural integer values of f_1 and f_2 satisfying (2.4).

2.1. Algebraic Method

For prime exponents $n \geq 3$, (2.4) with (1.4) can be written as

$$\sum_{i=0}^{\frac{n-3}{2}} \left[2^i S_{n-2(i+1)}^{(i+1)} \Delta^{i+1} \right] - \left(f_1 + 2nf_1f_2 + f_2 \right) = 0 \qquad ...(2.5)$$

The algebraic method involves simply finding the real roots of (2.5). As this equation is in the $(n-1)^{\text{th}}$ degree in the variable a, it has generally $(n-1)$ solutions, whose at least two are real of the general form $a = (\kappa + 1)$ and $a = -\kappa$, where κ are positive natural integers, if $(f_1 + 2nf_1f_2 + f_2)$ is a function Φ_n of triangular numbers $\Delta(\kappa)$ specific for each odd prime exponent n,

$$f_1 + 2nf_1f_2 + f_2 = \Phi_n(\Delta(\kappa)) \qquad ...(2.6)$$

As the bases a must be natural integers $a \geq 2$, the second solution $a = -\kappa$ is discarded.

Obviously, from (2.4), as $Q_n(a)$ is an integer function of $\Delta(a-1)$, one has

$$Q_n(\Delta(a-1)) = \Phi_n(\Delta(\kappa)) \qquad ...(2.7)$$

meaning that the two functions Q_n and Φ_n are identical with the variables κ and a such that

$$a = (\kappa + 1) \qquad ...(2.8)$$

As f_1 and f_2 can be permuted by symmetry, the solutions in a of (2.6) come for couples of f_1 and f_2 that comply with (2.6). Starting with a first value of f_1, the values of f_2 are searched in increasing order $f_{2,i}$, with i natural integers, such that the couples $(f_1, f_{2,i})$ comply with (2.6). For these couples $(f_1, f_{2,i})$, one finds increasing values of κ, noted κ_i, to which corresponds the solution

$$a_i = (\kappa_i + 1) \qquad ...(2.9)$$

All solutions in a of (2.5) in the form (2.9) are found once all couples of f_1 and f_2 complying with (2.6) are found. However, as we have a double infinity of possible values of f_1 and f_2, this method is not really practical. Instead, the solutions in a of (2.5) can be found by the following general method, where the solutions in a are calculated and presented differently.

2.2. Excluded f_i values

Before introducing the general method, we recall the notion of excluded values. Certain values of f_1 do not yield solutions for f_2 and a, simply because for these values of f_1, there are no values of f_2 such that relation (2.6) holds. These values of f_1 and f_2 are excluded values and can be calculated *a priori* by the method demonstrated in Lemma 1 in (Pletser, 2024a), allowing them to be skipped in all further methods and algorithms. It was found that the general form of excluded values of f_i is

$$f_i \not\equiv (\alpha + k\beta)\left(\bmod\left(2nk + \gamma\right)\right) \qquad ...(2.10)$$

for all integers $k \geq 0$, all integers α, and all odd integers β, γ complying with

$$2n\alpha = \beta\gamma - 1 \qquad ...(2.11)$$

except that $k \neq 0$ for $\alpha = 0$, and if $\beta = 1$ and $\gamma = 1$, k must be of the form

$$k = 2nuv + u\varepsilon + v\delta + r \qquad ...(2.12)$$

with u, v, δ, ε, r natural integers and $\delta\varepsilon = 2nr + 1$ with both δ and $\varepsilon \neq 0$ and $\neq 1$. In the general case of $\alpha \neq 0$ (i.e., β and $\gamma \neq \pm 1$), the excluded values f_i (2.10) are to be found for all non-negative integers k and for those positive and/or negative integer values of β and $\gamma \neq 1$ such that their product would equal $(2n\alpha + 1)$ with α either positive or negative, i.e., α and Υ are the positive and/or negative integer factors of the multiples of $2n$ augmented or diminished by a unity. If $(2n\alpha + 1)$ for $\alpha > 0$ (or $(2n\alpha - 1)$ for $\alpha < 0$) is itself a prime, then obviously no values of β and $\gamma \neq 1$ would be found and other values of α have to be considered. In practice, this method to find the excluded values is rather fast as it is sufficient to verify conditions (2.11) for the first integer values of α and k, as in most cases, similar excluded values are found repetitively, which allows also to ignore the special case of $\alpha = 0$, $\beta = 1$, $\gamma = 1$.

From (2.2), the forbidden forms of factors c_i corresponding to the excluded values f_i (2.10) are then, for all integers k,

$$c_i \not\equiv 0\left(\bmod\left(2nk - 1\right)\right) \text{ for } k > 0 \qquad ...(2.13)$$

$$c_i \not\equiv 0\left(\bmod\left(2nk \pm t\right)\right) \text{ for } k \geq 0 \qquad ...(2.14)$$

for all odd natural integers t, $1 < t < n$. These forbidden forms of factors c_i are always composites and the product of at least two factors, which are multiple of integers in the form $(2nj - 1)$ and/or $(2nj \pm t)$ with j natural integer and at least once $j = k$.

2.3. General Method

To introduce the general method, the following theorems are demonstrated, using (2.2) in (2.5).

Theorem 1: For all prime integer odd exponents n and for all factors $c_i = 2nf_i + 1$ of a $GM_{a,n}$ with f_i non-excluded integer values, the solutions in a of

$$\sum_{i=0}^{\frac{n-3}{2}} \left[2^i\, S_{n-2(i+1)}^{(i+1)}\, \Delta^{i+1}\right] - \left(f_1 + c_1 f_2\right) = 0 \qquad ...(2.15)$$

come periodically with cycles of length c_1 for a fixed value of the integer coefficient f_1 and for increasing values of f_2 satisfying (2.15).

Proof: Let f_1 be a first fixed value of the non-excluded integer coefficient appearing in the factor c_1. Let $f_{2,a}$ be the smallest non-negative value of the other non-excluded integer coefficient f_2 such that $(f_1 + c_1 f_{2,a})$ complies with (2.15). A first initial solution in a, noted a_a, can then be found such that (2.15) is satisfied, i.e.,

$$Q_n(\Delta(a_a - 1)) = f_1 + c_1 f_{2,a} \qquad ...(2.16)$$

For the same fixed value of f_1, other solutions in a, noted a_b, are found for other values of f_2, noted $f_{2,b}$, such that

$$Q_n(\Delta(a_b - 1)) = f_1 + c_1 f_{2,b} \qquad ...(2.17)$$

Noting δa the difference between a next solution a_b and the first solution a_a and δf_2 the difference between the corresponding integer coefficients f_2, i.e.,

$$\delta a = a_b - a_a \qquad ...(2.18)$$

$$\delta f_2 = f_{2,b} - f_{2,a} \qquad ...(2.19)$$

these differences δa and δf_2 are deduced from (2.17), with (2.18) and (2.19),

$$Q_n(\Delta(a_a + \delta a - 1)) = f_1 + c_1(f_{2,a} + \delta f_2) \qquad ...(2.20)$$

The triangular number of a sum is easily expressed as

$$\Delta((a_a - 1) + \delta a) = \Delta(a_a - 1) + a_a\delta a + \Delta(\delta a - 1)$$

$$= \Delta(a_a - 1) + \frac{\delta a(2a_a - 1 + \delta a)}{2} \qquad (2.21)$$

Replacing in the left term of (2.20) then yields, with (1.4),

$$Q_n = \left(\Delta(a_a - 1) + \frac{\delta a(2a_a - 1 + \delta a)}{2}\right)$$

$$= \sum_{i=0}^{\frac{n-3}{2}} \left[2^i S_{n-2(i+1)}^{(i+1)} \left(\Delta(a_a - 1) + \frac{\delta a(2a_a - 1 + \delta a)}{2}\right)^{i+1} \right] \qquad ...(2.22)$$

Developing the polynomial in the right term of (2.22) yields

$$\sum_{i=0}^{\frac{n-3}{2}} \left[2^i S_{n-2(i+1)}^{(i+1)} \left(\left(\Delta(a_a - 1)\right)^{i+1} + \sum_{j=0}^{i} \left[C_{j+1}^{i+1} \left(\Delta(a_a - 1)\right)^{i-j} \left(\frac{\delta a(2a_a - 1 + \delta a)}{2}\right)^{j+1} \right] \right) \right] \qquad ...(2.23)$$

where C_{j+1}^{i+1} is the binomial coefficient. By separating the first sums, one has

$$\sum_{i=0}^{\frac{n-3}{2}} \left[2^i S_{n-2(i+1)}^{(i+1)} \left(\Delta(a_a - 1)\right)^{i+1} \right]$$

$$+ \sum_{i=0}^{\frac{n-3}{2}} \left[2^i S_{n-2(i+1)}^{(i+1)} \sum_{j=0}^{i} \left[C_{j+1}^{i+1} \left(\Delta(a_a - 1)\right)^{i-j} \left(\frac{\delta a(2a_a - 1 + \delta a)}{2}\right)^{j+1} \right] \right] \qquad ...(2.24)$$

and, by factoring by $\left(\frac{\delta a(2a_a - 1 + \delta a)}{2}\right)$ and inverting the sums in the second sum term, one has

$$\sum_{i=0}^{\frac{n-3}{2}} \left[2^i S_{n-2(i+1)}^{(i+1)} \left(\Delta(a_a - 1)\right)^{i+1} \right]$$

$$+ \left(\frac{\delta a(2a_a - 1 + \delta a)}{2}\right) \sum_{j=0}^{\frac{n-3}{2}} \left[A_j \left(\delta a(2a_a - 1 + \delta a)\right)^j \right] \qquad ...(2.25)$$

where the integer coefficients A_j are integer functions of $\Delta(a_a - 1)$ and n,

$$A_j = \sum_{i=j}^{\frac{n-3}{2}} \left[S_{n-2(i+1)}^{(i+1)} C_{j+1}^{i+1} \left(2\Delta(a_a - 1)\right)^{i-j} \right] \qquad ...(2.26)$$

with obviously $A_{\frac{n-3}{2}}$ always equal to 1 for any value of n.

The first sum term in (2.25) is $Q_n(\Delta(a_a - 1))$ while the second one is noted $\delta Q_n(\Delta(a_a - 1))$. Replacing (2.25) in (2.20) and simplifying with (2.16), one has

$$\delta Q_n(\Delta(a_a - 1)) = c_1 \Delta f_2 \qquad ...(2.27)$$

Replacing with the second sum term of (2.25) yields

$$\delta f_2 = \frac{\left(\frac{\delta a(2a_a - 1 + \delta a)}{2}\right)\sum_{j=0}^{\frac{n-3}{2}}\left[A_j\left(\delta a(2a_a - 1 + \delta a)\right)^j\right]}{c_1} \qquad ...(2.28)$$

This relation means that the difference δf_2 between a next value and the first value of f_2 is a function of the difference δa between a next solution a_b and the first solution a_a, of the initial solution a_a itself and, of course, of the exponent n.

As the difference δf_2 must be integer, one of the factors in the right term of (2.28) must be divisible by c_1 depending on the value of δa.

Let the first factor δa of (2.28) be divisible by c_1, then

$$\delta a(r) = rc_1 \qquad ...(2.29)$$

where δa is noted $\delta a(r)$, with r positive integers. To this value of $\delta a(r)$ corresponds a difference $\delta f_2(r)$

$$\delta f_2(r) = \left(\frac{r(rc_1 + 2a_a - 1)}{2}\right)\sum_{j=0}^{\frac{n-3}{2}}\left[A_j\left(rc_1(rc_1 + 2a_a - 1)\right)^j\right] \qquad ...(2.30)$$

Relation (2.29) yields, from (2.18),

$$a_b = a_a + \delta a(r) = a_a + rc_1 \qquad ...(2.31)$$

meaning that for every initial solution a_a, there is an infinite number of other solutions a_b that can be found for all positive integers r and for couples of non-excluded integer values (f_1, f_{2b}) with

$$f_{2b} = f_{2a} + \delta f_2 \qquad ...(2.32)$$

and δf_2 given by (2.30). As any solution in a can be chosen as an initial solution a_a, all solutions corresponding to the initial couple of non-excluded integer values (f_1, f_{2a}) come periodically with cycles of length c_1.

Theorem 2: For all prime odd exponents n, for all factors $c_i = (2nf_i + 1)$ of a $GM_{a,n}$ with f_i non-excluded integers, and for a fixed value of f_1, all M solutions in a of (2.15) smaller than kc_1 can be ordered in pairs of solutions embedded into each others and the two solutions of each pair are related by

$$a_{M-j+1} + a_j = kc_1 + 1 \qquad ...(2.33)$$

for positive integers k and j, with $1 \leq j \leq M$.

Proof: Let a_1 be the first smallest solution in a of (2.15) corresponding to the smallest values of non-excluded integer coefficients f_1 and $f_{2,1}$ such that $(f_1 + c_1 f_{2,1})$ complies with (2.15) and let the following solutions a_j be ordered by their integer index j such that

$$a_1 < a_2 < ... < a_j < ... \qquad ...(2.34)$$

for the fixed value of f_1 and for increasing values of non-excluded integer coefficients $f_{2,j}$ such that $(f_1 + c_1 f_{2,j})$ complies with (2.25).

Let the second factor of (2.28) $(2a_a - 1 + \delta a)$ be divisible by c_1, then

$$\delta a(s) = sc_1 - 2a_a + 1 \qquad ...(2.35)$$

where δa is noted $\delta a(s)$, with s positive integers. To this value of $\delta a(s)$ corresponds a difference $\delta f_2(s)$

$$\delta f_2(s) = \left(\frac{s(sc_1 - 2a_a + 1)}{2}\right)\sum_{j=0}^{\frac{n-3}{2}}\left[A_j\left(sc_1(sc_1 - 2a_a + 1)\right)^j\right] \qquad ...(2.36)$$

Relation (2.35) means, from (2.18), that there is another solution a_b corresponding to a_a

$$a_b = a_a + \delta a(s) = a_a + sc_1 - 2a_a + 1 = sc_1 - a_a + 1 \tag{2.37}$$

As in Theorem 1, any solution in a could be taken as initial value a_a, this relation (2.37) holds for all solutions $a_j < sc_1$ and in particular for $s = k$.

As Theorem 1 showed that to all solutions in a correspond other solutions in a found by adding a multiple of c_1, we start first with $k = 1$ in (2.33) and $s = 1$ in (2.37).

Let m be the number of solutions $a_j < c_1$, with $1 \le j \le m$. As the smallest value that a_1 can take is 2, one has from (2.34)

$$a_1 < a_2 < \ldots < a_j < \ldots < a_{m-j+1} < \ldots < a_{m-1} < a_m < c_1 \qquad \text{...(2.38)}$$

Obviously, m must be even as there are $\frac{m}{2}$ pairs of solutions (a_j, a_q) such that, from (2.37),

$$a_j + a_q = c_1 + 1 \qquad \text{...(2.39)}$$

with j and q integers, with $1 \le j \le m$ and $1 \le q \le m$. If m would be odd, there would be a solution a_u, with $1 \le u \le m$, left unpaired that would be paired with another solution a_v, where either $a_v < a_1$, which makes no sense, or $a_v > a_m$, for which relation (2.39) would not hold.

If $a_a = a_1$ in (2.37), then obviously $a_b = a_m$ for relation (2.37) with $s = 1$ to hold, i.e.,

$$a_1 + a_m = c_1 + 1 \qquad \text{...(2.40)}$$

and there are $\frac{m}{2}$ pairs of solutions (a_j, a_{m-j+1}) such that

$$a_j + a_{m-j+1} = c_1 + 1 \qquad \text{...(2.41)}$$

These $\frac{m}{2}$ pairs of solutions (a_j, a_{m-j+1}) are embedded into each others as shown by (2.38).

For $k > 1$, let M be the number of solutions $a_j < kc_1$, with $1 \le j \le M$, with

$$1 < a_1 < a_2 < \ldots < a_j < \ldots < a_{M-j+1} < \ldots < a_{M-1} < M < kc_1 \qquad \text{...(2.42)}$$

From Theorem 1, adding rc_1 repetitively to each solution a of the inequalities (2.38) for integers r from 1 to $(k-1)$ yields new solutions that are ordered similarly to (2.38)

$$\begin{aligned} 1 &< a_1 < \ldots < a_m < c_1 < a_{m+1} = (a_1 + c_1) < \ldots < a_{2m} = (a_m + c_1) < 2c_1 \qquad \text{...(2.43)} \\ &< a_{2m+1} = (a_1 + 2c_1) < \ldots < a_{3m} = (a_m + 2c_1) < 3c_1 < \ldots < (k-1)c_1 \\ &< a_{(k-1)m+1} = (a_1 + (k-1)c_1) < \ldots < a_{km} = (a_m + (k-1)c_1) < kc_1 \end{aligned}$$

which is identical to (2.42), with $M = km$ showing that M is even and that (2.33) holds from (2.37) with $s = k$.

Note that, as the smallest value that a_1 can take is 2, (2.40) yields that $a_m \le (c_1 - 1)$ and that the first $\frac{m}{2}$ solutions a_j are such that

$$1 < a_1 < a_2 < \ldots < a_{\frac{m}{2}} < \frac{c_1 + 1}{2} \qquad \text{...(2.44)}$$

The number m of solutions in a smaller than c_1 is given in the following theorem in function of the exponent n.

Theorem 3: For all prime odd exponents n and for all factors $c_i = (2nf_i + 1)$ of a $GM_{a,n}$ with f_i non-excluded integers, $m = (n-1)$, i.e., the solutions in a of (2.15) come periodically in an infinite number of groups of $\frac{(n-1)}{2}$ pairs of solutions in a with cycles of length c_1 for a fixed value of the integer coefficient f_1 and for increasing values of f_2 satisfying (2.15).

Proof: For the simple case of $n = 3$, the theorem is easily demonstrated as there will be only one pair of solutions in the first group of solutions, i.e., $m = n - 1 = 2$, and one has from (2.40)

$$a_1 + a_2 = c_1 + 1 \qquad ...(2.45)$$

From Theorem 1, all other solutions in a can then be found by adding multiples of c_1 to these first two solutions.

For prime $n > 3$, let $a_a = a_1$ in (2.28) and let the polynomial in δa in (2.28) be divisible by c_1, then

$$\sum_{j=0}^{\frac{n-3}{2}} \left[A_j \left(\delta a \left(2a_1 - 1 + \delta a \right) \right)^j \right] = tc_1 \qquad ...(2.46)$$

with t positive integers. This polynomial of degree $\frac{(n-3)}{2}$ in the variable (δa $(2a_1 - 1 + \delta a)$) has $\frac{(n-3)}{2}$ solutions whose at least two are real as n is odd. The real solutions are functions $F_{\left(\frac{n-3}{2}\right)}$ depending on n, the triangular number of $(a_1 - 1)$ and the product tc_1, such that

$$\delta a \left(2a_1 - 1 + \delta a \right) = F_{\left(\frac{n-3}{2}\right)} \left(\Delta \left(a_1 - 1 \right), \, tc_1 \right) = F_{\left(\frac{n-3}{2}\right)} \left(a_1, \, tc_1 \right) \qquad ...(2.47)$$

yielding

$$\delta a \left(t^* \right) = \frac{-\left(2a_1 - 1 \right) + \sqrt{\left(2a_1 - 1 \right)^2 + 4F_{\left(\frac{n-3}{2}\right)} \left(a_1, \, t^* c_1 \right)}}{2} \qquad ...(2.48)$$

where, as δa must be a positive integer, one considers only the positive sign in front of the root sign and only those integers t, noted t^*, yielding positive integers δa, noted $\delta a(t^*)$, and the function $F_{\left(\frac{n-3}{2}\right)}$ (a_1, t^*c_1) must take positive values for all integers t^*.

From (2.28) with (2.47) and (2.46), the difference $\delta f_2(t^*)$ can then be calculated as

$$\delta f_2 \left(t^* \right) = \frac{t^* F_{\left(\frac{n-3}{2}\right)} \left(a_1, \, t^* c_1 \right)}{2} \qquad ...(2.49)$$

Let the integers t^* yielding positive integers $\delta a(t)$ in (2.48) be noted in increasing order t_j^* with integers $j > 1$.

One has to show that $m = (n - 1)$, i.e., that there is $(n - 1)$ solutions in a smaller than c_1, or

$$a_1 < a_2 < ... < a_{n-2} < a_{n-1} < c_1 < a_n < a_{n+1} < ... \qquad ...(2.50)$$

As, from (2.18), one has

$$a_j = a_1 + \delta a \left(t_{j-1}^* \right) \qquad ...(2.51)$$

for $2 \leq j \leq (n - 2)$, and with (2.40) and (2.31) with $a_a = a_1$, (2.50) yields

$$\begin{aligned} a_1 < a_2 &= a_1 + \delta a \left(t_1^* \right) < a_3 = a_1 + \delta a \left(t_2^* \right) < \ldots \\ &< a_{n-3} = a_1 + \delta a \left(t_{n-4}^* \right) < a_{n-2} = a_1 + \delta a \left(t_{n-3}^* \right) \\ &< a_{n-1} = a_1 + \delta a \left(s = 1 \right) = c_1 - a_1 + 1 < c_1 \\ &< a_n = a_1 + \delta a \left(r = 1 \right) = a_1 + c_1 < a_{n+1} = a_1 + \delta a \left(t_{n-2}^* \right) < \ldots \end{aligned} \qquad ...(2.52)$$

From this, four conditions are deduced on values of $\delta a(t^*)$ (2.48) for $m = (n-1)$ to hold.

The first condition is that the first difference $\delta a\left(t_1^*\right)$ must be positive

$$\delta a\left(t_1^*\right) > 0 \qquad ...(2.53)$$

which is the case if the function $F_{\left(\frac{n-3}{2}\right)}\left(a_1, t^* c_1\right)$ is always positive. This first condition constrains also the integer t_1^* if δa is replaced in the polynomial (2.46) by 0, i.e., considering only the terms for $j = 0$ in (2.46), it yields

$$t_1^* > \frac{A_0}{c_1} = \frac{\sum_{i=0}^{\frac{n-3}{2}}\left[C_i^{n-(i+2)}\left(2\Delta\left(a_a - 1\right)\right)^i\right]}{c_1} \qquad ...(2.54)$$

with

$$C_i^{n-(i+2)} = S_{n-2(i+1)}^{(i+1)} C_1^{i+1} \qquad ...(2.55)$$

The second condition is that all differences $\delta a\left(t_j^*\right)$ must increase monotonically for increasing j, i.e.,

$$\delta a\left(t_1^*\right) < \delta a\left(t_2^*\right) < \ldots < \delta a\left(t_{n-4}^*\right) < \delta a\left(t_{n-3}^*\right) \qquad ...(2.56)$$

which is the case as the functions $F_{\left(\frac{n-3}{2}\right)}\left(a_1, t^* c_1\right)$ (2.47) involve in all generality root functions of $\Delta(a_1 - 1)$ and of the product t^*c_1, and increases also monotonically for increasing t_j^* in the product $t_j^* c_1$ as all terms in the polynomial (2.46) are positive.

The third condition is that the difference $\delta a\left(t_{n-3}^*\right)$ must be smaller than the difference $\delta a(s)$ (2.35) for $s = 1$, i.e.,

$$\delta a\left(t_{n-3}^*\right) < c_1 - 2a_1 + 1 \qquad ...(2.57)$$

or

$$F_{\left(\frac{n-3}{2}\right)}\left(a_1, t_{n-3}^* c_1\right) < c_1\left(c_1 - 2a_1 + 1\right) \qquad ...(2.58)$$

This third condition constrains also the integer t_{n-3}^* if δa is replaced in the polynomial (2.46) by $(c_1 - 2a_1 + 1)$, yielding

$$t_{n-3}^* < \frac{\sum_{j=0}^{\frac{n-3}{2}}\left[A_j\left(c_1\left(c_1 - 2a_1 + 1\right)\right)^j\right]}{c_1} \qquad ...(2.59)$$

The fourth condition is that the difference $\delta a\left(t_{n-2}^*\right)$ must be greater than the difference $\delta a(r)$ (2.29) for $r = 1$, i.e.,

$$\delta a\left(t_{n-2}^*\right) > c_1 \qquad ...(2.60)$$

or

$$F_{\left(\frac{n-3}{2}\right)}\left(a_1, t_{n-2}^* c_1\right) > c_1\left(c_1 + 2a_1 - 1\right) \qquad ...(2.61)$$

This fourth condition constrains also the integer t_{n-2}^* if Δa is replaced in the polynomial (2.46) by c_1, yielding

$$t_{n-2}^{*} > \frac{\sum_{j=0}^{\frac{n-3}{2}}\left[A_j\left(c_1\left(c_1+2a_1-1\right)\right)^j\right]}{c_1} \quad ...(2.62)$$

If these four conditions are verified, then relation (2.52) holds.

As from Theorem 2, the solutions in a can be paired, the differences $\delta a\left(t_j^*\right)$ and the functions $F_{\left(\frac{n-3}{2}\right)}\left(a_1, t_j^* c_1\right)$ are related two by two as follows. From (2.41) and with

$$\delta a\left(t_{j-1}^*\right) = a_j - a_1 \quad ...(2.63)$$

all differences $\delta a\left(t_{j-1}^*\right)$ are such that

$$\delta a\left(t_{j-1}^*\right) + \delta a\left(t_{n-1-j}^*\right) = c_1 - 2a_1 + 1 \quad ...(2.64)$$

for positive integers $2 \le j \le \frac{(n-1)}{2}$, which means that the functions $F_{\left(\frac{n-3}{2}\right)}\left(a_1, t_j^* c_1\right)$ (2.47) are linked by (2.64) such that

$$F_{\left(\frac{n-3}{2}\right)}\left(a_1, t_{n-3-j}^* c_1\right) = F_{\left(\frac{n-3}{2}\right)}\left(a_1, t_{j-1}^* c_1\right) - c_1\sqrt{\left(2a_1-1\right)^2 + 4F_{\left(\frac{n-3}{2}\right)}\left(a_1, t_{j-1}^* c_1\right) + \left(c_1\right)^2} \quad ...(2.65)$$

Therefore, the first $(n-1)$ solutions can be grouped in $\frac{(n-1)}{2}$ pairs, where the first solution is associated to the $(n-1)^{\text{th}}$ solution, i.e., (a_1, a_{n-1}), where both solutions are separated by the difference δa $(s = 1)$ (2.35); the second solution is paired with the $(n-2)^{\text{th}}$ solution, i.e., (a_2, a_{n-2}), etc., and the $\left(\frac{n-1}{2}\right)^{\text{th}}$ with the $\left(\frac{n+1}{2}\right)^{\text{th}}$ solutions, i.e., $\left(a_{\frac{n-1}{2}}, a_{\frac{n+1}{2}}\right)$.

To show the indefinite periodicity with cycles of length c_1 of groups of $(n-1)$ solutions, all groups of $(n-1)$ solutions are numbered by positive integers k, starting with $k = 1$ for the first group.

From (2.29) and (2.35), the first and last solutions of the k^{th} group of $(n-1)$ solutions are such that

$$a_{(k-1)(n-1)+1} = a_1 + \delta a(r) = a_1 + (k-1)c_1 \quad ...(2.66)$$

$$a_{k(n-1)} = a_1 + \delta a(s) = -a_1 + 1 + kc_1 = a_{n-1} + (k-1)c_1 \quad ...(2.67)$$

for all positive integers k and with $r = (k-1)$ in (2.29) and $s = k$ in (2.35).

For the other $(n-3)$ solutions of the k^{th} group, the first $(n-3)$ integers t_j^* that yield positive $\delta a\left(t_j^*\right)$ (2.48) are such that

$$\delta a\left(t_{(k-1)(n-3)+j}^*\right) + \delta a\left(t_{j-1}^*\right) + (k-1)c_1 \quad ...(2.68)$$

yielding

$$a_{(k-1)(n-1)+j} = a_1 + \delta a\left(t_{(k-1)(n-3)+j}^*\right) = a_1 + \delta a\left(t_{j-1}^*\right) + (k-1)c_1 = a_j + (k-1)c_1 \quad ...(2.69)$$

for positive integers $2 \le j \le (n-2)$ and for all positive integers k.

Alternatively, as the first seed solution a_a can be chosen arbitrarily such as (2.16) holds, and, for example taking as new seed any $a_b = a_a + \delta a\left(t_{a-1}^*\right)$ of the first group, by (2.29) and with $r = (k-1)$, the corresponding value in the k^{th} group is $a_c = a_b + (k-1)c_1 = a_a + \delta a\left(t_{a-1}^*\right) + (k-1)c_1$.

This yields generally

$$a_{(k-1)(n-1)+1}=\left(a_1+(k-1)c_1\right)<a_{(k-1)(n-1)+2}=\left(a_1+\delta a\left(t_1^*\right)+(k-1)c_1\right)$$

$$<\ldots<a_{k(n-1)-1}=\left(a_1+\delta a\left(t_{n-3}^*\right)+(k-1)c_1\right)$$

$$<a_{k(n-1)}=\left(a_1+\delta a(s=1)+(k-1)c_1\right)=kc_1-a_1+1 \quad ...(2.70)$$

This shows that all solutions of the first group repeat themselves with cycles of length c_1 and that in any group all solutions are ordered similarly as in the first group, and, from (2.66) to (2.68), the $(n-1)$ solutions in any following group can be grouped in $\frac{(n-1)}{2}$ pairs embedded into each other.

Theorem 3 yields the following two corollaries on sums and differences of solutions in a within a group of $(n-1)$ solutions.

Corollary 4: For all prime odd exponents n and for all factors $c_i = 2nf_i + 1$ of a $GM_{a,n}$ with f_i non-excluded integer values, within the k^{th} group of $\frac{(n-1)}{2}$ pairs of solutions a, the sum of the two solutions of each pair is such that

$$a_{(k-1)(n-1)+j}+a_{k(n-1)-(j-1)}=(2k-1)c_1+1 \quad ...(2.71)$$

for all positive integers k and j with $1 \le j \le \frac{(n-1)}{2}$.

Proof: For $j = 1$, the proof is immediate as, from (2.66), (2.67) and (2.40) with $m = (n-1)$, one has

$$a_{(k-1)(n-1)+1}+a_{k(n-1)}=a_1+a_{n-1}+2(k-1)c_1=(2k-1)c_1+1 \quad ...(2.72)$$

For $2 \le j \le \frac{(n-1)}{2}$, from (2.69) and (2.41) with $m = (n-1)$, one has

$$a_{(k-1)(n-1)+j}+a_{k(n-1)-(j-1)}=a_j+a_{(n-1)-(j-1)}+2(k-1)c_1=(2k-1)c_1+1 \quad ...(2.73)$$

Corollary 5: For all prime odd exponents n and for all factors $c_i = 2nf_i + 1$ of a $GM_{a,n}$ with f_i non-excluded integer values, within the k^{th} group of $\frac{(n-1)}{2}$ pairs of solutions a, the differences of successive solutions in a are such that

$$a_{(k-1)(n-1)+j+1}-a_{(k-1)(n-1)+j}=a_{k(n-1)-j+1}-a_{k(n-1)-j} \quad ...(2.74)$$

for all positive integers k and j with $1 \le j \le \frac{(n-1)}{2}$.

Proof: The proof is immediate as, from (2.71),

$$a_{(k-1)(n-1)+j}+a_{k(n-1)-(j-1)}=a_{(k-1)(n-1)+j+1}+a_{k(n-1)-j}=(2k-1)c_1+1 \quad ...(2.75)$$

yields immediately (2.74).

Note that this second corollary shows as well that the differences $\delta a\left(t_j^*\right)$ in Theorem 3 proof have also the following property on their differences

$$\delta a\left(t_1^*\right)=\delta a(s=1)-\delta a\left(t_{n-3}^*\right)=c_1-2a_1+1-\delta a\left(t_{n-3}^*\right) \quad ...(2.76)$$

$$\delta a\left(t_1^*\right)-\delta a\left(t_{j-1}^*\right)=\delta a\left(t_{n-3-j}^*\right)-\delta a\left(t_{n-4-j}^*\right) \quad ...(2.77)$$

for j positive integers and here $2 \le j \le \frac{(n-3)}{2}$.

To summarize, for a fixed non-excluded value of the integer coefficient f_1, all the solutions in a of (2.15) are found in groups of $\frac{(n-1)}{2}$ pairs of solutions from the first seed solution a_1 to which is added the first $\frac{(n-3)}{2}$ differences $\delta a\left(t_j^*\right)$ (2.48) for those first $\frac{(n-3)}{2}$ integer values t_j^* that yield positive integers $\delta a\left(t_j^*\right)$ in (2.48), then the remaining $\frac{(n-3)}{2}$ solutions of the first group can be found by (2.41) to form the first group of $\frac{(n-1)}{2}$ pairs of solutions. This first group is repeated indefinitely by adding the differences $\delta a(r = k-1)$ (2.29) for all positive integer k to all solutions of the first group. This is summarized in Table 1 where in the first group, a_1 is the first seed solution, the next $\frac{(n-3)}{2}$ solutions a_j are found by adding the differences $\delta a\left(t_{j-1}^*\right)$ determined by (2.48), the next $\frac{(n-1)}{2}$ solutions a_{n-j} are found by subtracting a_j from $(c_1 + 1)$, and in all following k^{th} group, all following solutions $a_{(k-1)(n-1)+j}$ are found by adding $(k-1)c_1$ to a_j.

Table 1: Values of a_j for First and k^{th} Groups of Solutions

First Group (k = 1)	k^{th} Group
a_1	$a_{(k-1)(n-1)+1} = a_1 + (k-1)c_1$
$a_2 = a_1 + \delta a\left(t_1^*\right)$	$a_{(k-1)(n-1)+2} = a_2 + (k-1)c_1$
…	…
$a_j = a_1 + \delta a\left(t_{j-1}^*\right)$	$a_{(k-1)(n-1)+j} = a_j + (k-1)c_1$
…	…
$a_{\frac{n-1}{2}} = a_1 + \delta a\left(t_{\frac{n-3}{2}}^*\right)$	$a_{(k-1)(n-1)+\frac{n-1}{2}} = a_{\frac{n-1}{2}} + (k-1)c_1$
$a_{\frac{n+1}{2}} = c_1 - a_{\frac{n-1}{2}} + 1$	$a_{(k-1)(n-1)+\frac{n+1}{2}} = a_{\frac{n+1}{2}} + (k-1)c_1$
…	…
$a_{n-j} = c_1 - a_j + 1$	$a_{(k-1)(n-1)+n-j} = a_{n-j} + (k-1)c_1$
…	…
$a_{n-2} = c_1 - a_2 + 1$	$a_{(k-1)(n-1)+n-2} = a_{n-2} + (k-1)c_1$
$a_{n-1} = c_1 - a_1 + 1$	$a_{(k-1)(n-1)+n-1} = a_{n-1} + (k-1)c_1$

All this can be summarized in a single relation

$$a_{(k-1)(n-1)+\frac{n\pm(1-2j)}{2}} = \frac{(1-c_1) \pm \left(2a_{\left(\frac{n+1}{2}-j\right)} - c_1 - 1\right)}{2} + kc_1 \qquad \text{...(2.78)}$$

for $1 \le j \le \frac{(n-1)}{2}$, giving all solutions in a in all k groups with k positive integers, where the + (respectively –) sign corresponds to the first (respectively second) solution in each of the $\frac{(n-1)}{2}$ pairs of solutions.

Replacing in δf_2 (2.28), δa by $\delta a(r) = (k-1)c_1$ and by $\delta a(s) = (kc_1 - 2a_1 + 1)$ yield respectively $\delta f_2(r)$ and $\delta f_2(s)$

$$\delta f_2(r) = \frac{(k-1)\left((k-1)c_1 + 2a_1 - 1\right)}{2} \sum_{j=0}^{\frac{n-3}{2}} \left[A_j \left((k-1)c_1\left((k-1)c_1 + 2a_1 - 1\right)\right)^j \right] \qquad \text{...(2.79)}$$

$$\delta f_2(s) = \frac{k(kc_1 - 2a_1 + 1)}{2} \sum_{j=0}^{\frac{n-3}{2}} \left[A_j \left(kc_1 (kc_1 - 2a_1 + 1) \right)^j \right] \quad ...(2.80)$$

This means that all solutions in a of (2.15) for the first fixed value of f_1 are characterized once the first solution a_1 is found. The problem of determining all the values of the bases a that render a $GM_{a,n}$ composite or prime (if f_2 is nil) for a specific prime exponent n is reduced to find the first seed solution a_1 and to determine the first $\frac{(n-3)}{2}$ differences $\delta a(t_j^*)$ by (2.48), for a given value of f_1 and then start again for all other non-excluded values of f_1 with (2.15).

2.4. Practical Methods

In practice, one does not have to calculate for all values of f_1. It is sufficient to start with the values

$$f_1 = Q_n(\Delta(a-1)) \quad ...(2.81)$$

for all bases a. For these values, one would obviously have $f_2 = 0$ and if

$$c_1 = 2nf_1 + 1 = 2nQ_n(\Delta(a-1)) + 1 \quad ...(2.82)$$

cannot be further decomposed in product of factors similar to (2.2), the corresponding $GM_{a,n}$ number is obviously prime. The following values of f_2 corresponding to this first value of f_1 are found for values of a at intervals $c_1 = 2nf_1 + 1$ by (2.9). These other values of f_2 are then taken as new values of f_1, different from other solutions of the function $Q_n(\Delta(a-1))$, and the process can start again. Examples of this algorithm are given further for the cases $n = 3$ and 5. This algorithm allows to determine whether a $GM_{a,n}$ is composite and to calculate its main factors. Note that in the above algorithms, some integer values of f_2 will not be generated and are called excluded values. These correspond to excluded values of f_1 that can be calculated in advance by the method given in a lemma in (Pletser, 2024a) and in Section 2.2 Excluded f_i values, allowing to skip them in the above algorithms.

A simpler step by step algorithm to determine whether a $GM_{a,n}$ is prime or composite and to determine its factors is as follows. Among all possible pairs of f_1 and f_2 that are solutions of (2.6), the pair $f_1 = Q_n(\Delta(a-1))$ and $f_2 = 0$ will always be a solution for values of a such that $2nf_1+1 = 2nQ_n(\Delta(a-1)) + 1 = GM_{a,n}$ is prime. To determine whether a $GM_{a,n}$ is composite or not, one has to verify that the factor c_1 corresponding to $f_1 = Q_n(\Delta(a-1))$ cannot be further decomposed with another pair of values of f_1 and f_2. For this, one forms from (2.6) and (1.4) the ratio.

$$R = \frac{Q_n(\Delta(a-1)) - i}{2ni + 1} \quad ...(2.83)$$

and one tests simply this ratio for all positive integers i until this ratio R is smaller than 1. If for a value of i, the ratio R becomes an integer (which in fact is f_2), the $GM_{a,n}$ corresponding to this value of a is composite and its main factors can be calculated with $f_1 = i$ and $f_2 = R$. If for all values of i until the ratio is smaller than 1, this ratio never becomes integer, the $GM_{a,n}$ corresponding to this value of a is prime. In practice, instead of all positive integers i, it is sufficient to test the ratio R for all integer factors of $Q_n(\Delta(a-1))$ as values of i, removing also the excluded values of f_1.

2.5. Where are the Primes?

Let's first recall that the primes $GM_{a,n}$ are to be found among those $GM_{a,n}$ having $f_1 = Q_n(\Delta(a-1))$ (2.81). Using a method similar to Erathostenes sieve, the following step by step algorithm allows to find the $GM_{a,n}$ primes among the first N $GM_{a,n}$ for $2 \le a \le N+1$.

Step 1: Determine the excluded values of f_1 like in Section 2.2 Excluded f_i values.

Step 2: For each non-excluded values of f_1, say f_1^*, and starting with the first (smallest) value, if there exists a^* such that

$$f_1^* = Q_n(\Delta(a^*-1)) \quad ...(2.84)$$

then the $GM_{a,n}$ corresponding to this a^* could be prime, as the corresponding f_2 is nil. To determine whether this $GM_{a^*,n}$ is prime, test the ratio (2.83) for natural integers i equal to non-excluded values f_1 smaller than f_1^*.

If this $GM_{a^*,n}$ is prime, then strike out all the multiples of this prime $GM_{a^*,n}$ which are found for

$$a_k = kc_1^* + a^* \quad ...(2.85)$$

$$a_k = kc_1^* - a^* + 1 \quad ...(2.86)$$

with $c_1^* = 2n f_1^* + 1$ and k positive natural integers.

If this $GM_{a^*,n}$ is not prime (i.e., if the ratio R (2.83) is integer, say R^*, for $i = i^*$), it is obviously composite with $f_1 = i^*$ and $f_2 = R^*$. Then strike out this $GM_{a^*,n}$ and all the multiples of $c_1^* = 2ni^* + 1$ and of $c_2^* = 2nR^* + 1$ which are found for

$$a_k = kc_i^* + a^* \quad ...(2.87)$$

$$a_k = kc_i^* - a^* + 1 \quad ...(2.88)$$

with c_i^* being respectively c_1^* and c_2^*.

Repeat Step 2 until the last non-excluded value $f_1^* \le Q_n\left(\Delta\left(N\right)\right)$.

Step 3: For those values of f_1 that are not equal to $Q_n(\Delta(a-1))$ for any a, in most of the cases these f_1 will appear as f_2 for one of the f_1^* considered in Step 2. Once this $f_1^* \ne Q_n\left(\Delta\left(a^* - 1\right)\right)$ has appeared as a f_2, the corresponding a, say a^*, yields a composite $GM_{a^*,n}$. Then strike out this $GM_{a^*,n}$ and all the multiples of $c_1^* = 2n f_1^* + 1$ which are found for (2.85) and (2.86).

Repeat Step 3 until the last non-excluded value $f_1^* < Q_n\left(\Delta\left(N\right)\right)$.

When Step 3 is completed, the remaining unstruck values of $GM_{a,n}$ are the primes less than $GM_{(N+1),n}$.

If the natural integer values of $Q_n(\Delta(a-1))$ are easy to calculate, then consider first those values of f_1 that are equal to $Q_n(\Delta(a-1))$ for some a. This algorithm will be illustrated for $n = 3$.

3. Results and Discussion: Bases *a* for Composite $GM_{a,n}$ for Prime Exponents *n* = 3 to 17

3.1. Bases a *Yielding Generalized Mersenne Composites for n = 3*

3.1.1. Algebraic Method

For $n = 3$, relations (2.5) and (1.4), with

$$Q_3(a) = \Delta(a-1) \quad ...(3.1)$$

yield

$$a^2 - a - 2(f_1 + 6f_1f_2 + f_2) = 0 \quad ...(3.2)$$

which has in general the two real solutions

$$a = \frac{1 \pm \sqrt{1 + 8\left(f_1 + 6f_1f_2 + f_2\right)}}{2} \quad ...(3.3)$$

yielding positive integers a for the + sign in (3.3) and if the discriminant is the square of an odd integer, i.e., if

$$f_1 + 6f_1f_2 + f_2 = \Delta(K) \quad ...(3.4)$$

with K positive integers, which corresponds to relation (2.7)

$$Q_3(a) = \Delta(a-1) = \Phi_3(\Delta(K)) = \Delta(K) \quad ...(3.5)$$

that gives the first integer solution a of the form (2.8), $a = K + 1$.

3.1.2. Excluded f_i Values

Certain values of f_1 do not yield solutions in a and f_2, simply because for these values of f_1, there are no values of f_2 such that relation (3.5) can hold. These values of f_1 and f_2 are excluded values and are $f_i \neq 4, 9, 14, 19, 20, 24, 29, 31, 42, 48, 53, 65, ...$

These can be calculated *a priori*, allowing them to be skipped in above algorithms.

From (2.10), the single general relation of excluded f_i values is obtained for the integer triplet $(\alpha, \beta, \gamma) = (0, -1, -1)$ and is

$$f_i \not\equiv -k\left(\bmod\left(6k-1\right)\right) \quad ...(3.6)$$

for all positive integers k, yielding for

$$k = 1: f_i \not\equiv -1\left(\bmod 5\right) \text{ or } f_i \not\equiv 4\left(\bmod 5\right) \text{ i.e., } f_i \neq 4, 9, 14, 19, 24, \ldots \quad ...(3.7)$$

$$k = 2: f_i \not\equiv -2\left(\bmod 11\right) \text{ or } f_i \not\equiv 9\left(\bmod 11\right) \text{ i.e., } f_i \neq 9, 20, 31, 42, 53, \ldots \quad ...(3.8)$$

$$k = 3: f_i \not\equiv -3\left(\bmod 17\right) \text{ or } f_i \not\equiv 14\left(\bmod 17\right) \text{ i.e., } f_i \neq 14, 31, 48, 65, 82, \ldots \quad ...(3.9)$$

$$k = 4: f_i \not\equiv -4\left(\bmod 23\right) \text{ or } f_i \not\equiv 19\left(\bmod 23\right) \text{ i.e., } f_i \neq 19, 42, 65, 88, 111, \ldots \quad ...(3.10)$$

etc. All other positive and negative integer values of the triplet (α, β, γ) complying with (2.11) would yield repetitively similar excluded values. The excluded values f_i (3.6) with (2.2) yield the forbidden forms of factors c_i (2.13), with positive integers k,

$$c_i \not\equiv 0\left(\bmod\left(6k-1\right)\right) \quad ...(3.11)$$

These forbidden forms of factors c_i are always composites and the product of at least two factors, which are multiple of integers in the form $(6j - 1)$ with j integers and with at least once $j = k$. For $f_i \neq 4, 9, 14, 19, 20, 24, 29, 31, 42, 48, 53, 65, ...$, it yields successively $c_i \neq 25, 55, 85, 115, 121, 145, 175, 187, 253, 289, 319, 391, ...$

3.1.3. General Method

From (2.45), one has two solutions in the first group such as

$$a_1 < a_2 = (c_1 - a_1 + 1) \quad ...(3.12)$$

that repeats itself with cycles c_1, as (2.70) reduces to

$$a_{2k-1} = (a_1 + (k-1)c_1) < a_{2k} = (kc_1 - a_1 + 1) \quad ...(3.13)$$

These two relations for a_{2k-1} and a_{2k} can be combined into a single general relation (2.78),

$$a_{\left(2k-\left(\frac{1\pm1}{2}\right)\right)} = \frac{\left(1-c_1\right) \pm \left(2a_1 - c_1 - 1\right)}{2} + kc_1 \quad ...(3.14)$$

yielding all values of solutions a_i knowing a_1 for all positive integers k.

The expression of δf_2 (2.28) reduces to

$$\delta f_2 = \frac{\delta a\left(2a_1 - 1 + \delta a\right)}{2c_1} \quad ...(3.15)$$

The values of $f_{2,(2k-1)}$ and $f_{2,2k}$ for a_{2k-1} and a_{2k} can be found in function of a_1, f_1, c_1 and k, from (2.19) with (2.2), and with (2.79) or (2.80)

$$f_{2,(2k-1)} = f_{2,1} + \delta f_2(r) = \left(\frac{a_1(a_1-1)-2f_1}{2c_1}\right) + \left(\frac{(k-1)\left((k-1)c_1+2a_1-1\right)}{2}\right) \quad ...(3.16)$$

$$f_{2,2k} = f_{2,1} + \delta f_2(s) = \left(\frac{a_1(a_1-1)-2f_1}{2c_1}\right) + \left(\frac{k(kc_1-2a_1+1)}{2}\right) \quad ...(3.17)$$

for all positive integers k. These two relations can be combined into a single one

$$f_{2,\left(2k-\left(\frac{1\pm 1}{2}\right)\right)} = \frac{1}{2}\left(\frac{c_1-1}{2} + \frac{(a_1-1)(a_1-c_1)-2f_1}{c_1} + k(k-1)c_1 \pm \frac{(2k-1)(2a_1-1-c_1)}{2}\right) \quad ...(3.18)$$

Note that for this case of $n = 3$, in addition to the relation (2.71)

$$a_{2k-1} + a_{2k} = (2k-1)c_1 + 1 \quad ...(3.19)$$

the relation

$$a_{2k-1} - a_{2k} = 2(f_{2,1} - f_{2,2}) \quad ...(3.20)$$

also holds for all positive integers k, yielding, for $k = 1$, the relations

$$a_1 = 3f_1 + f_{2,1} - f_{2,2} + 1;\ a_2 = 3f_1 - f_{2,1} + f_{2,2} + 1 \quad ...(3.21)$$

$$a_1 + a_2 = 6f_1 + 2 = c_1 + 1 \quad ...(3.22)$$

and that the relation

$$f_{2,(2k-1)} + f_{2,2k} = \frac{c_1-1}{2} + \frac{(a_1-1)(a_1-c_1)-2f_1}{c_1} + k(k-1)c_1 \quad ...(3.23)$$

from (3.16) and (3.17) holds for all couples of values $(f_{2,(2k-1)}, f_{2,2k})$ within a cycle of length c_1.

3.1.4. Examples

Following the algebraic method, starting with the first non-excluded value of $f_1 = 1$, a first solution from (3.4) is given by $f_2 = 0$ yielding $\Delta(K) = 1$ for $K = 1$, giving $a_1 = 2$ in (3.3) or (2.8). As $c_1 = 2nf_1 + 1 = 7$ cannot be further factorized, the $GM_{2,3}$ for $a = 2$ is prime. A second solution is found for $f_2 = 2$ yielding $\Delta(K) = 15$ in (3.4) for $K = 5$, giving $a = 6$. And so on.

Instead of sweeping all values of f_2 in search of K for (3.5) to hold, the above general method is followed. For $f_1 = 1$ (and $c_1 = 7$), the general relation (3.14) yields the first pair of solution for $k = 1$, $a_1 = 2$ and $a_2 = 6$. The corresponding values of f_2 can be found by (3.18), yielding here respectively $f_{2,1} = 0$ and $f_{2,2} = 2$. Other solutions for $k = 2, 3, \ldots$ are from (3.14) $a_{2k-1} = 9, 16, \ldots$ for $f_{2,(2k-1)} = 5, 17, \ldots$, and $a_{2k} = 13, 20, \ldots$ for $f_{2,2k} = 11, 27, \ldots$.

Solutions for other non-excluded values of f_1 are shown in Table 2 for $1 \leq f_1 \leq 10$ and $k = 1$ to 3, where for each value of f_1, the first and second lines correspond respectively to the first and second solutions of the solution pair. The corresponding composite $GM_{a,3}$ are shown in Table 2; other $GM_{a,3}$ can be found in Sequence A003215 in (Sloane, 2024).

Note that, when f_1 is a triangular number ($f_1 = 1, 3, 6, 10, \ldots$), (3.18) yields $f_{2,1} = 0$ as first solution for $k = 1$, which means that, if the factor c_1 cannot be further decomposed in similar factors (2.2), the corresponding Generalized Mersenne number is prime.

This simple approach explains why the first composite in the series of $GM_{a,n}$ for $n = 3$ in Table 3 appears for $a = 6$. The smallest non-nil values of f_1 and f_2 that yield integers a in (3.3) are $f_1 = 1$ and $f_2 = 2$, as $f_1 = f_2 = 1$ does not yield a triangular number in (3.5) and an integer a. Therefore all the $GM_{a,3}$ values for $a < 6$ cannot be composites and must be primes, as

Table 2: Values of a_κ and $f_{2,\kappa}$ Yielding Composite $GM_{a,3}$ (Except for $2 \le a_\kappa \le 5$)

f_1	c_1	$k = 1$			$k = 2$			$k = 3$		
		a_κ	$f_{2,\kappa}$	$GM_{a,3}$	a_κ	$f_{2,\kappa}$	$GM_{a,3}$	a_κ	$f_{2,\kappa}$	$GM_{a,3}$
1	7	**2**	0	**7**	9	5	217	16	17	721
		6	2	91	13	11	469	20	27	1141
2	13	6	1	91	19	13	1027	32	38	2977
		8	2	169	21	16	1261	34	43	3367
3	19	**3**	0	**19**	22	12	1387	41	43	4921
		17	7	817	36	33	3781	55	78	8911
5	31	9	1	217	40	25	4681	71	80	14911
		23	8	1519	54	46	8587	85	115	21421
6	37	**4**	0	**37**	41	22	4921	78	81	18019
		34	15	3367	71	67	14911	108	156	34669
7	43	17	3	817	60	41	10621	103	122	31519
		27	8	2107	70	56	14491	113	147	37969
8	49	23	5	1519	72	52	15337	121	148	43561
		27	7	2107	76	58	17101	125	158	46501
10	61	**5**	0	**61**	66	35	12871	127	131	48007
		57	26	9577	118	113	41419	179	261	95587

Note: Where $\kappa = (2k - 1)$ and $\kappa = 2k$ for the first and second lines for each value of f_1. Note that 4 and 9 are excluded values of f_1. $GM_{a,3}$ in bold are primes.

Table 3: Decomposition of $GM_{a,3}$

a	$GM_{a,3} = 2 \cdot 3 \cdot \Delta + 1$	
2	$7 = 2 \cdot 3 \cdot 1 + 1$	prime
3	$19 = 2 \cdot 3 \cdot 3 + 1$	prime
4	$37 = 2 \cdot 3 \cdot 6 + 1$	prime
5	$61 = 2 \cdot 3 \cdot 10 + 1$	prime
6	$91 = 2 \cdot 3 \cdot 15 + 1$	$= 7 \cdot 13 = (2 \cdot 3 + 1)(2^2 \cdot 3 + 1)$
7	$127 = 2 \cdot 3 \cdot 21 + 1$	prime
8	$169 = 2 \cdot 3 \cdot 28 + 1$	$= 13 \cdot 13 = (2^2 \cdot 3 + 1)(2^2 \cdot 3 + 1)$
9	$217 = 2 \cdot 3 \cdot 36 + 1$	$= 7 \cdot 31 = (2 \cdot 3 + 1)(2 \cdot 15 + 1)$
10	$271 = 2 \cdot 3 \cdot 45 + 1$	prime

can be seen from Table 2 where $a = 2$ (for $f_1 = 1$), $a = 3$ (for $f_1 = 3$), $a = 4$ (for $f_1 = 6$), $a = 5$ (for $f_1 = 10$) yield primes as $f_2 = 0$ for these cases.

Note further that, as the roles of f_1 and f_2 can be permuted, certain solutions are found twice.

3.1.5. Practical Methods

In practice, one does not have to calculate for all values of f_1. It is sufficient to start with all the values $f_1 = \Delta(a - 1)$ for all a; the other values of f_1 which are not triangular numbers are generated as f_2 corresponding to the first triangular numbers of f_1. Let's develop the following algorithm following Table 4, where prime (respectively composite) values of $GM_{a,n}$ are in bold (respectively italic) characters.

Starting with the first triangular number 1, we know that for $a = 2$, $GM_{2,3} = 7$ is prime, to which corresponds $f_1 = \Delta(a - 1) = 1$ and obviously $f_2 = 0$. The following values of f_2 for $f_1 = 1$ would appear, with positive integers k, from (3.13) respectively for $a_{2k-1} = 7(k-1) + 2 = 2, 9, 16, 23, \ldots$, and for $a_{2k} = 7k - 1 = 6, 13, 20, \ldots$, for which all $GM_{a,3}$ are composites

Table 4: Evolution of Values of $GM_{a,3}$, c_1, f_1 and f_2 for n = 3 and $2 \leq a \leq 50$

			$f_1=$	**1**	**2**	**3**	**5**	**6**	**7**	**8**	**10**
			$c_1=$	**7**	**13**	**19**	**31**	**37**	**43**	**49**	**61**
a	$GM_{a,3}$	Q_3	f_1,f_2	$f_{2\pm}$	$f_{2\pm}$	$f_{2\pm}$	$f_{2\pm}$	$f_{2\pm}$	$f_{2\pm}$	$f_{2\pm}$	$f_{2\pm}$
2	**7**	1	1, 0	0_+	-	-	-	-	-	-	-
3	**19**	3	3, 0	-	-	0_+	-	-	-	-	-
4	**37**	6	6, 0	-	-	-	-	0_+	-	-	-
5	**61**	10	10, 0	-	-	-	-	-	-	-	0_+
6	*91*	15	1, 2	2_-	1_+	-	-	-	-	-	-
7	**127**	21	21, 0	-	-	-	-	-	-	-	-
8	*169*	28	2, 2	-	2_-	-	-	-	-	-	-
9	*217*	36	1, 5	5_+	-	-	1_+	-	-	-	-
10	**271**	45	45, 0	-	-	-	-	-	-	-	-
11	**331**	55	55, 0	-	-	-	-	-	-	-	-
12	**397**	66	66, 0	-	-	-	-	-	-	-	-
13	*469*	78	1, 11	11_-	-	-	-	-	-	-	-
14	**547**	91	91, 0	-	-	-	-	-	-	-	-
15	**631**	105	105, 0	-	-	-	-	-	-	-	-
16	*721*	120	1, 17	17_+	-	-	-	-	-	-	-
17	*817*	136	3, 7	-	-	7_-	-	-	3_+	-	-
18	**919**	153	153, 0	-	-	-	-	-	-	-	-
19	*1027*	171	2, 13	-	13_+	-	-	-	-	-	-
20	*1141*	190	1, 27	27_-	-	-	-	-	-	-	-
21	*1261*	210	2, 16	-	16_-	-	-	-	-	-	-
22	*1387*	231	3, 12	-	-	12_+	-	-	-	-	-
23	*1519*	253	1, 36	36_+	-	-	8_-	-	-	5_+	-
24	**1657**	276	276, 0	-	-	-	-	-	-	-	-
25	**1801**	300	300, 0	-	-	-	-	-	-	-	-
26	**1951**	325	325, 0	-	-	-	-	-	-	-	-
27	*2107*	351	1, 50	50_-	-	-	-	-	8_-	7_-	-
28	**2269**	378	378, 0	-	-	-	-	-	-	-	-
29	**2437**	406	406, 0	-	-	-	-	-	-	-	-
30	*2611*	435	1, 62	62_+	-	-	-	-	-	-	-
31	**2791**	465	465, 0	-	-	-	-	-	-	-	-
32	*2977*	496	2, 38	-	38_+	-	-	-	-	-	-
33	**3169**	528	528, 0	-	-	-	-	-	-	-	-
34	*3367*	561	1, 80	80_-	43_-	-	-	15_-	-	-	-
35	**3571**	595	595, 0	-	-	-	-	-	-	-	-
36	*3781*	630	3, 33	-	-	33_-	-	-	-	-	-
37	*3997*	666	1, 95	95_+	-	-	-	-	-	-	-
38	**4219**	703	703, 0	-	-	-	-	-	-	-	-
39	**4447**	741	741, 0	-	-	-	-	-	-	-	-
40	*4681*	780	5, 25	-	-	-	25_+	-	-	-	-
41	*4921*	820	1, 117	117_-	-	43_+	-	22_+	-	-	-
42	**5167**	861	861, 0	-	-	-	-	-	-	-	-

Table 4 (Cont.)

			$f_1=$	**1**	**2**	**3**	**5**	**6**	**7**	**8**	**10**
			$c_1=$	**7**	**13**	**19**	**31**	**37**	**43**	**49**	**61**
a	$GM_{a,3}$	Q_3	f_1, f_2	$f_{2\pm}$	$f_{2\pm}$	$f_{2\pm}$	$f_{2\pm}$	$f_{2\pm}$	$f_{2\pm}$	$f_{2\pm}$	$f_{2\pm}$
43	**5419**	903	903, 0	-	-	-	-	-	-	-	-
44	*5677*	946	1, 135	135_+	-	-	-	-	-	-	-
45	*5941*	990	2, 76	-	76_+	-	-	-	-	-	-
46	**6211**	1035	1035, 0	-	-	-	-	-	-	-	-
47	*6487*	1081	2, 83	-	83_-	-	-	-	-	-	-
48	*6769*	1128	1, 161	161_+	-	-	-	-	-	-	-
49	**7057**	1176	1176, 0	-	-	-	-	-	-	-	-
50	**7351**	1225	1225, 0	-	-	-	-	-	-	-	-

Notes: $GM_{a,3}$ in bold (respectively italic) characters are prime (respectively composite), $Q_3 = \Delta(a-1)$, subscripts + and – refer to f_{2+} and f_{2-}.

(except of course pour the first seed value $a_1 = 2$ for $k = 1$). The corresponding values of f_2 can be calculated by (3.18) and (3.23) and are respectively 5, 17, 36, ... and 2, 11, 27, ...

These values of f_2 are taken as new values of f_1 for which new c_1 are calculated and new pairs of f_1 and f_2 are found yielding other composites $GM_{a,3}$.

For example, taking $f_1 = 2$, yielding $c_1 = 13$, gives the first value of $a = 6$ for $f_1 = 2$ and $f_2 = 1$, yielding the composite $GM_{6,3} = 91$. Other composites are found like above for $a_{2k-1} = 13k + 6 = 19, 32, 45, ...$, and for $a_{2k} = 13k - 5 = 8, 21, 34, ...$, for which all $GM_{a,3}$ are composites. These new values of f_2 are taken as new values of f_1 and the process can start again.

One proceeds similarly for other triangular numbers 3, 6, 10, ... as seeds for $f_1 = \Delta(a-1)$.

This algorithm allows to determine whether a $GM_{a,3}$ is composite and to calculate its main factors.

In the simpler step by step algorithm, among all possible pairs of f_1 and f_2 solutions of (3.5), the pair $f_1 = \Delta(a-1)$ and $f_2 = 0$ will always be a solution for a such as $(2nf_1 + 1)$ is prime. To determine whether a $GM_{a,3}$ is composite or not, one verifies that the factor corresponding to $f_1 = \Delta(a-1)$ cannot be further decomposed with another pair of values of f_1 and f_2, testing the ratio

$$R = \frac{\Delta(a-1) - i}{6i + 1} \qquad ...(3.24)$$

for all positive integers i until this ratio $R < 1$. If for a value of i, the ratio R becomes a positive natural integer (which in fact is f_2), the $GM_{a,3}$ corresponding to this a is composite and its main factors can be calculated with $f_1 = i$ and $f_2 = R$. If for all i until $R < 1$, this ratio R never becomes integer, the $GM_{a,3}$ corresponding to this a is prime.

Let's develop the following example.

Starting with $a = 2$, and $i = 1$, one has $R = 0 < 1$, thus $GM_{a,3} = 7$ is prime.

For $a = 3$ and $i = 1$, one has $R = \frac{2}{7} < 1$, thus $GM_{3,3} = 19$ is prime.

For $a = 4$ and $i = 1$, one has $R = \frac{5}{7} < 1$, thus $GM_{4,3} = 37$ is prime.

For $a = 5$ and $i = 1$ and 2, one has successively $R = \frac{9}{7} > 1$; $R = \frac{8}{13} < 1$, thus $GM_{5,3} = 61$ is prime.

For $a = 6$ and $i = 1$, one has $R = \frac{14}{7} = 2$, thus $GM_{6,3} = 91$ is composite.

For $a = 7$ and $i = 1$, 2 and 3, one has successively $R = \frac{20}{7} > 1$; $R = \frac{19}{13} > 1$; $R = \frac{18}{19} < 1$, thus $GM_{7,3} = 127$ is prime.

For $a = 8$ and $i = 1$ and 2, one has successively $R = \frac{27}{7} > 1$; $R = \frac{26}{13} = 2$, thus $GM_{8,3} = 169$ is composite.

And so on.

3.1.6. Where are the $GM_{a,3}$ Primes ?

We follow the algorithm of Section 2.5 with $Q_3(\Delta(a-1)) = \Delta(a-1)$ to find the $GM_{a,3}$ primes for $2 \le a \le 50$ (i.e., $N = 49$). Multiples of $GM_{a,3}$ primes will be found for values of $a_{k+} = c_1^* k + a^*$ and $a_{k-} = c_1^* k - a^* + 1$ with k positive natural integers, $c_1^* = 2nf_1^* + 1 = 6f_1^* + 1$ and a^* the first solution in a of (3.2) with $f_1 = f_1^* = \Delta(a^* - 1)$ and $f_2 = 0$. The first step needs to determine the maximum values of a and f_1 such that the smallest value of a_k (here a_{k-}) is smaller than 50 for $k = 1$.

Step 1: Determine the largest values $a_{\max}$ of a and $f_{1\max}$ of f_1 such that $f_{1\max} = \Delta(a_{\max} - 1)$ and $a_{k-} \le 50$ for $k = 1$, i.e.,

$$a_{1-} = c_{1\max} - a_{\max} + 1 = 6f_{1\max} - a_{\max} + 2 \quad ...(3.25)$$

$$= 6\Delta(a_{\max} - 1) - a_{\max} + 2 \quad ...(3.26)$$

$$= 3a_{\max}(a_{\max} - 1) - a_{\max} + 2 \le 50 \quad ...(3.27)$$

yielding $a_{\max} < 4.721$ or $a_{\max} = 4$ and $f_{1\max} = \Delta(a_{\max} - 1) = 6$.

Step 2: Determine the excluded values of $f_1 < f_{1\max}$, i.e., $f_1 \ne 4$ (see Section 3.1.2).

Step 3: For the non-excluded values of $f_1 \le f_{1\max}$, i.e., $f_1 = 1, 2, 3, 5, 6$, select those that equal to $\Delta(a-1)$ for some values of a, i.e., that are triangular numbers $f_1 = 1, 3, 6$.

Start with the smallest $f_1^* = 1$, one has that for $a^* = 2$, $\Delta(a^* - 1) = 1 = f_1^*$ and test the ratio R (3.24), but as there are no values of natural integers i smaller than $f_1^* = 1$, $a^* = 2$ yields that $GM_{2,3} = c_1^* = 6f_1^* + 1 = 7$ is prime. Then strike all the $GM_{a,3}$ multiples of 7 which are found for $a_{k+} = 7k + 2 < 50$ and $a_{k-} = 7k - 1 < 50$ with k positive natural integers, i.e., for $a_{k+} = 9, 16, 23, 30, 37, 44$ and $a_{k-} = 6, 13, 20, 27, 34, 41, 48$.

For the next value of $f_1^* = \Delta(a^* - 1) = 3$ for $a^* = 3$, and testing the ratio R (3.24) for $i = 1$, $R = \frac{2}{7} < 1$, it yields that $GM_{3,3} = c_1^* = 6f_1^* + 1 = 19$ is prime.

Then strike all the $GM_{a,3}$ multiples of 19, found for $a_{k+} = 19k + 3 < 50$ and $a_{k-} = 19k - 2 < 50$, i.e., for $a_{k+} = 22, 41$ and $a_{k-} = 17, 36$.

The last value of $f_1^* = \Delta(a^* - 1) = 6$ for $a^* = 4$, and $R = \frac{5}{7} < 1$, yielding that $GM_{4,3} = c_1^* = 6f_1^* + 1 = 37$ is prime. Then strike all the multiples of $GM_{4,3}$ appearing for $a_{k+} = 37k + 4 < 50$ and $a_{k-} = 37k - 3 < 50$, i.e., for $a_{k+} = 41$ and $a_{k-} = 34$.

Stop Step 3 here as the next value of $f_1^* = \Delta(a^* - 1) = 10$ for $a^* = 5$ would yield $c_1^* = 6f_1^* + 1 = 61$ and $a_{k+} = 61k + 5$ and $a_{k-} = 61k - 4$ which are both larger than 50 for positive integers k.

Step 4: Select the non-excluded values of f_1 which are not triangular numbers.

For $f_1^* = 2$, determine the smallest natural integer value of f_2 and the corresponding value of a such that (3.2) holds. Here obviously $f_2 = 1$ and $a^* = 6$, which yields that $GM_{6,3}$ is composite as already found in Step 2. Then strike out all the

$GM_{a,3}$ which are multiple of $c_1^* = 6f_1^* + 1 = 13$ which appear for $a_{k+} = 13k + 6 < 50$ and $a_{k-} = 13k - 5 < 50$, i.e., for $a_{k+} = 19$, 32, 45 and $a_{k-} = 8, 21, 34, 47$.

For $f_1^* = 5$, one finds similarly $f_2 = 1$ and $a^* = 9$, which yields that $GM_{9,3}$ is composite as already found in Step 2. Then strike out all the $GM_{a,3}$ which are multiple of $c_1^* = 6f_1^* + 1 = 31$ which appear for $a_{k+} = 31k + 9 < 50$ and $a_{k-} = 31k - 8 < 50$, i.e., for $a_{k+} = 40$ and $a_{k-} = 23$.

Stop Step 4 here as all the following non-excluded value of $f_1^* \neq \Delta(a^* - 1)$ yield composite $GM_{a,3}$ which are already eliminated in Step 3. For example, the value $f_1^* = 7$ would yield $f_2 = 3$ and $a^* = 17$ giving $c_1^* = 6f_1^* + 1 = 43$ and $a_{k+} = 43k + 17 > 50$ for positive integers k and $a_{k-} = 43k - 16$ which, for $k = 1$, is 27 which is already eliminated in Step 3 and then a_{k-} are larger than 50 for integers $k > 1$.

This algorithm is relatively fast and allows to find in four steps (including 3 substeps in Step 3 and 2 substeps in Step 4) where the 26 $GM_{a,3}$ primes and the 23 $GM_{a,3}$ composites are for $2 \leq a \leq 50$ (see Table 4), without having to calculate the values of all the $GM_{a,3}$. Only five values of $Q_3(\Delta(a-1)) = \Delta(a-1)$ had to be calculated for the first solutions a^*.

3.2. Bases a Yielding Generalized Mersenne Composites for n = 5

3.2.1. Algebraic Method

For $n = 5$, (2.5) and (1.4), with

$$Q_5(a) = \Delta(a-1)\,(2\Delta(a-1) + 1) \qquad \text{...(3.28)}$$

yield

$$a^4 - 2a^3 + 2a^2 - a - 2(f_1 + 10f_1f_2 + f_2) = 0 \qquad \text{...(3.29)}$$

which has the general solutions

$$a = \frac{1 \pm \sqrt{1 + 2\left(-1 \pm \sqrt{1 + 8\left(f_1 + 10f_1f_2 + f_2\right)}\right)}}{2} \qquad \text{...(3.30)}$$

yielding two complex solutions of no interest here, and two real solutions. One of these will be a positive integer solution for the positive signs in front of the two square root signs, and if the discriminant under the first square root sign is the square of an odd integer, $(2K + 1)^2 = 1 + 8\Delta(K)$, i.e., if

$$f_1 + 10f_1f_2 + f_2 = \Delta(2\Delta(K)) = \Delta(K)\,(2\Delta(K) + 1) \qquad \text{...(3.31)}$$

with K positive integers, which corresponds to relation (2.7)

$$Q_5(a) = \Delta(a-1)\,(2\Delta(a-1) + 1) = \Phi_5(\Delta(K)) = \Delta(K)\,(2\Delta(K) + 1) \qquad \text{...(3.32)}$$

that gives the first integer solution in a of the form (2.8), $a = K + 1$.

The triangular number of twice a triangular number of K is the sum of two pyramidal numbers, or the product of a triangular number by its double augmented by one unity, or the triple sum of octahedral numbers from 1 to K (Conway and Guy, 1996) or twice the sum of the first K^{th} cubes plus the sum of the first K^{th} integers i.

$$\Delta(2\Delta(K)) = 3\sum_{i=1}^{K}\left[Oct_i\right] = \sum_{i=1}^{K}\left[\frac{i\left(2i^2 + 1\right)}{3}\right] = 2\sum_{i=1}^{K}\left[i^3\right] + \sum_{i=1}^{K}\left[i\right] \qquad \text{...(3.33)}$$

i.e., for integers $K = 1$ to 5, $\Delta(2\Delta(K))$ takes the values 3, 21, 78, 210, 465, etc.

Only for these values does a in (3.30) take integer values greater than 1, which are further determined by the general method.

3.2.2. Excluded f_i Values

As previously, certain integer values of f_i do not yield integer solutions for a and f_2, namely $f_1 \neq 2, 5, 8, 9, 11, 14, \dots$. From (2.10), the general relation of excluded values f_i is

$$f_i \not\equiv (\alpha + k\beta) \left(\bmod\left(10k + \Upsilon\right)\right) \quad \text{...(3.34)}$$

for all positive integers k, and for integers $\alpha \neq 0, \beta \neq 1$ and $\Upsilon \neq 1$ complying with (2.11).

For the integer triplet $(\alpha, \beta, \Upsilon) = (0, -1, -1)$, (3.34) gives a first general relation of excluded values f_i

$$f_i \not\equiv -k \left(\bmod\left(10k + \Upsilon\right)\right) \quad \text{...(3.35)}$$

for all positive integers k, yielding for

$$k = 1 : f_i \not\equiv -1 \left(\bmod 9\right) \text{ or } f_i \not\equiv 8\left(\bmod 9\right) \quad \text{...(3.36)}$$

$$k = 2 : f_i \not\equiv -2 \left(\bmod 19\right) \text{ or } f_i \not\equiv 17\left(\bmod 19\right) \quad \text{...(3.37)}$$

$$k = 3 : f_i \not\equiv -3 \left(\bmod 29\right) \text{ or } f_i \not\equiv 26\left(\bmod 29\right) \quad \text{...(3.38)}$$

etc. For $(\alpha, \beta, \Upsilon) = \left(-1, \pm 3, \mp 3\right)$, a second general relation of excluded values f_i is

$$f_i \not\equiv \left(\pm 3k - 1\right) \left(\bmod\left(10k \mp 3\right)\right) \quad \text{...(3.39)}$$

for all non-negative integers k, yielding respectively for the upper and lower signs

$$k = 0 : f_i \not\equiv -1 \left(\bmod 3\right) \text{ or } f_i \not\equiv 2\left(\bmod 3\right) \quad \text{...(3.40)}$$

$$k = 1 : f_i \not\equiv 2 \left(\bmod 7\right) \text{ and } f_i \not\equiv -4\left(\bmod 13\right) \text{ or } f_i \not\equiv 9\left(\bmod 13\right) \quad \text{...(3.41)}$$

$$k = 2 : f_i \not\equiv 5 \left(\bmod 17\right) \text{ and } f_i \not\equiv -7\left(\bmod 23\right) \text{ or } f_i \not\equiv 16\left(\bmod 23\right) \quad \text{...(3.42)}$$

$$k = 3 : f_i \not\equiv 8 \left(\bmod 27\right) \text{ and } f_i \not\equiv -10\left(\bmod 33\right) \text{ or } f_i \not\equiv 23\left(\bmod 33\right) \quad \text{...(3.43)}$$

etc. For all other positive and negative integer values of α, β and Υ complying with (2.11), the general expressions of f_i will not be different. From (2.2), forbidden forms of factors c_i (2.20) corresponding to excluded values f_i (3.35) and (3.39) are of the form, with positive integers k,

$$c_i \not\equiv 0 \left(\bmod\left(10k - 1\right)\right) \text{ for } k > 0 \quad \text{...(3.44)}$$

$$c_i \not\equiv 0 \left(\bmod\left(10k \pm 1\right)\right) \text{ for } k \geq 0 \quad \text{...(3.45)}$$

These forbidden forms of factors c_i are always composites and the product of at least two factors, which are multiple of integers of the form $10j - 1$ and/or $10j \pm 3$, with j integers and with at least once $j = k$.

3.2.3. General Method

From (2.70), the first pair of solutions (a_{4k-3}, a_{4k}) in all k groups is

$$a_{4k-3} = a_1 + (k - 1)c_1 < a_{4k} = kc_1 - a_1 + 1 \quad \text{...(3.46)}$$

From (2.46), with the coefficient (2.26)

$$A_0 = 4\Delta(a_1 - 1) + 1 = 2a_1(a_1 - 1) + 1 \quad \text{...(3.47)}$$

the function $F_{(1)}(a_1, t^*c_1)$ is

$$F_{(1)}(a_1, t^*c_1) = t^*c_1 - 2a_1(a_1 - 1) - 1 \quad \text{...(3.48)}$$

yielding

$$\delta a\left(t^*\right)=\frac{-\left(2a_1-1\right)+\sqrt{4t^*c_1-\left(2a_1-1\right)^2-2}}{2} \quad \text{...(3.49)}$$

which takes positive integer values first, for the positive sign in front of the root sign and if the square root is greater than the first term of (3.49) yielding

$$t^*>\frac{2a_1\left(a_1-1\right)+1}{c_1} \quad \text{...(3.50)}$$

which verifies the first condition (2.54), and second, for those values of integers t^* that yield positive integers value to $\delta a(t^*)$, i.e., if the discriminant in (3.49) is the square of an odd integer, i.e.

$$4t^*c_1-(2a_1-1)^2-2=(2T+1)^2 \quad \text{...(3.51)}$$

with T positive integers, yielding

$$t^*=\frac{T\left(T+1\right)+a_1\left(a_1-1\right)+1}{c_1}=\frac{2\Delta\left(T\right)+2\Delta\left(a_1-1\right)+1}{c_1} \quad \text{...(3.52)}$$

The first two smallest integer values of T, i.e., T_1 and T_2 such that t^* in (3.52) is integer and complies with (3.50) give t_1^* and t_2^*, yielding $\delta a\left(t_1^*\right)$ and $\delta a\left(t_2^*\right)$. These two values of $\delta a(t^*)$ are related by (2.64) and $\delta a\left(t_2^*\right)$ can be found from $\delta a\left(t_1^*\right)$ as

$$\delta a\left(t_2^*\right)=c_1-2a_1+1-\delta a\left(t_1^*\right)=c_1-a_1-a_2+1 \quad \text{...(3.53)}$$

yielding, from (2.47) and (3.48),

$$t_2^*=t_1^*+c_1-\left(2a_1+\delta a\left(t_1^*\right)\right)+1=t_1^*+c_1-2a_2+1 \quad \text{...(3.54)}$$

Note also that $\delta a\left(t_2^*\right)$ in (3.53) verifies the condition (2.57) as $a_1 < a_2$.

The fourth condition (2.62) yields that

$$t_3^*>\frac{\left(c_1+a_1\right)\left(c_1+a_1-1\right)+a_1\left(a_1-1\right)+1}{c_1} \quad \text{...(3.55)}$$

constraining the value of T_3 from (3.52) to

$$T_3 > c_1 + a_1 - 1 \quad \text{...(3.56)}$$

The first group of two pairs of solutions reads then

$$a_1 < a_2 = a_1 + \delta a\left(t_1^*\right) < a_3 = a_1 + \delta a\left(t_2^*\right) = c_1a_2 + 1 < a_4 = c_1 - a_1 + 1 \quad \text{...(3.57)}$$

The other groups of two pairs of solutions are found by adding $(k-1)c_1$ to these first four solutions as

$$\begin{aligned} &a_{4k-3} = a_1 + (k-1)\,c_1 < a_{4k-2} = a_2 + (k-1)c_1 \\ &\qquad < a_{4k-1} = a_3 + (k-1)c_1 < a_{4k} = a_4 + (k-1)c_1 \end{aligned} \quad \text{...(3.58)}$$

for all positive integers k. These relations for a can be combined into the single general relation (2.78), yielding all values of solutions a

$$a_{4k-\frac{3\pm(2j-1)}{2}}=\frac{\left(1-c_1\right)\pm\left(2a_{3-j}-c_1-1\right)}{2}+kc_1 \quad \text{...(3.59)}$$

for $1 \le j \le 2$ and for all positive integers k.

The differences $\delta f_2(r)$, $\delta f_2(s)$ and $\delta f_2(t^*)$ are found from (2.79), (2.80) and (2.49) and yield the general relation for f_2

$$f_{2,\left(4k-\frac{3\pm(2j-1)}{2}\right)} = \left(\frac{1}{2}\right)\left[\left(\frac{c_1-1}{2}\right)\left(c_1\left(c_1-1\right)+1\right)\right.$$

$$+\left(\frac{\left(a_{3-j}-1\right)\left(a_{3-j}-c_1\right)\left(\left(a_{3-j}-1\right)\left(a_{3-j}-c_1\right)+c_1\left(2c_1-1\right)+1\right)-2f_1}{c_1}\right)$$

$$+k\left(k-1\right)c_1\left(k+\left(k-1\right)\left(k\left(k-1\right)c_1^2+6\left(a_{3-j}-1\right)\left(a_{3-j}-c_1\right)+c_1\left(2\left(c_1-1\right)-1\right)\right)\right)$$

$$\left.\pm\left(2k-1\right)\left(2a_{3-j}-c_1-1\right)\left(k\left(k-1\right)c_1^2+\left(a_{3-j}-1\right)\left(a_{3-j}-c_1\right)+\left(\frac{c_1\left(c_1-1\right)+1}{2}\right)\right)\right] \quad \text{...(3.60)}$$

for $1 \le j \le 2$ and for all positive integers k.

3.2.4. Examples of Calculation

From the algebraic method, starting with the first non-excluded value of $f_1 = 1$, yielding $c_1 = 11$, one has that the smallest value of f_2 that satisfies (3.31) is $f_{2,1} = 7$ to which corresponds $K = 3$ as, from (3.31),

$$c_1 f_{2,1} + f_1 = 11 \cdot 7 + 1 = \Delta(2\Delta(K=3)) = 78 \quad \text{...(3.61)}$$

yielding from (2.8) the first seed solution $a_1 = 4$ for this value of $f_1 = 1$. From (3.49), with $T = 4$ and $t_1^* = 3$ in (3.52), $\delta a\left(t_1^*\right) = 1$ and $a_2 = 5$. From (3.59), the other solutions for $k = 1$ are $a_3 = 7$ and $a_4 = 8$. From (3.60), the values of f_2 are respectively 19, 82, and 145. Other values are given in Table 5 for $1 \le f_1 \le 10$ and $k = 1$ to 3. For the next non-excluded value of $f_1 = 3$ (which is the value of $\Delta(2\Delta(K))$ for $K = 1$) yielding $c_1 = 31$, one has $f_{2,1} = 0$ in (3.31), meaning that $a_1 = 2$ does not yield a composite but a Generalized Mersenne prime, which is the Mersenne prime $M_5 = 31$. From (3.49), with $T = 9$ and $t_1^* = 3$ in (3.52), $\delta a\left(t_1^*\right) = 8$ and $a_2 = 10$. From (3.59), the other solutions for $k = 1$ are $a_3 = 22$ and $a_4 = 30$. Corresponding composite $GM_{a,5}$ can be found in Sequence A022521 in (Sloane, 2024).

3.2.5. Practical Method

Like in Section 3.1.2, one does not have to calculate for all values of f_1 and one can start with all the values f_1 from (3.31) with $f_2 = 0$

$$f_1 = \Delta(a-1)\,(2\Delta(a-1)+1) \quad \text{...(3.62)}$$

for all a. The other values of f_1 which are not appearing in (3.62) are generated as f_2 values corresponding to the first value of f_1 (3.62). Like above, let's follow the algorithm in Table 6, where prime (respectively composite) values of $GM_{a,n}$ are in bold (respectively italic) characters.

Instead of starting with the first non-excluded value of $f_1 = 1$, we start with the first value of $a_1 = 2$ yielding from (3.62) $f_1 = 3$ to which corresponds $c_1 = 31$, which cannot be further decomposed and is prime, meaning that $f_1 = 3$ and $f_{2,1} = 0$ are genuine coefficients of the factors c_1 and c_2. One obtains then the prime $GM_{2,5} = 31$. The following values of $f_{2,\left(4k-\frac{3\pm3}{2}\right)}$ on the positive and negative branches for $f_1 = 3$ appear respectively for $a_{4k-3} = 2 + kc_1 = 31k + 2$ for all positive integers k, that is, for $k = 2$ to 4, $a_{4k-3} = 33$ (for $f_2 = 18003$), 64, 95, ..., and for $a_{4k} = -(2 - 1 - kc_1) = 31k - 1$ that is, for $k = 2$ to 4, $a_{4k} = 30$ (for $f_2 = 12222$), 61, 92, ..., for which all $GM_{a,5}$ are composites. Still for $f_1 = 3$, the second solution is $a_2 = 10$, to which corresponds $f_{2,2} = 132$ from (3.60). The following positive and negative branches of other values of $f_{2,\left(4k-\frac{3\pm1}{2}\right)}$ for $f_1 = 3$

Table 5: Values of a_κ and $f_{2,\kappa}$ Yielding Composite $GM_{a,5}$ for $n = 5$

		$k = 1$		$k = 2$		$k = 3$	
f_1	c_1	a_κ	$f_{2,\kappa}$	a_κ	$f_{2,\kappa}$	a_κ	$f_{2,\kappa}$
1	11	4	7	15	2014	26	19234
		5	19	16	2629	27	22432
		7	82	18	4270	29	30007
		8	145	19	5332	30	34444
3	31	2	0	33	18003	64	262275
		10	132	41	43407	72	421575
		22	3450	53	122553	84	784125
		30	12222	61	216117	92	1130625
4	41	9	64	50	73231	91	818101
		12	214	53	92662	94	932089
		30	9241	71	301291	112	1884964
		33	13612	74	355939	115	2096161
6	61	16	474	77	280752	138	2929965
		24	2502	85	417924	146	3673689
		38	16215	99	771627	160	5305074
		46	35139	107	1054527	168	6452190
7	71	4	1	75	216958	146	3156268
		19	826	90	451888	161	4673263
		53	53509	124	1638301	195	10078498
		68	146209	139	2591326	210	13566028
10	101	27	2443	128	1308286	229	13495768
		29	3268	130	1392325	231	13974505
		73	136786	174	4485943	275	28107415
		75	152515	176	4696390	277	28935568

Note: Where $\kappa = (4k-3)$, $(4k-2)$, $(4k-1)$ and $4k$ respectively from the first to the fourth lines for each value of f_1.

and $a_2 = 10$ appear respectively for $a_{4k-2} = 31k + 10$ for all positive integers k, that is, for $k = 2$ to 4, $a_{4k-2} = 41$ (for $f_2 = 43407$), 72, 103, ..., and for $a_{4k-1} = 31k - 9 = 22$ (for $f_2 = 3450$), 53, 84, ..., for which all $GM_{a,5}$ are again composites. All the corresponding values of f_2 are taken as new values of f_1 and the process is repeated for these values of f_1. Moving now to the next value of $a_1 = 3$ yields from (3.62) $f_1 = 21$ (not represented in Table 6) and $f_{2,1} = 0$. As $c_1 = 211$ cannot be further decomposed and is prime, one obtains the prime $GM_{3,5} = 211$. The following values of $f_{2,\left(4k-\frac{3\pm3}{2}\right)}$ on the positive and negative branches for $f_1 = 21$ appear respectively for $a_{4k-3} = 3 + kc_1 = 211k + 3$ for all positive integers k, i.e., $a_{4k-3} = 214$, 425, 636, ..., and for $a_{4k} = -(3 - 1 - kc_1) = 211k - 2$, i.e., $a_{4k} = 209$, 420, 631, ..., for which all $GM_{a,5}$ are composites. Still for $f_1 = 21$, from (3.49), with $T = 87$ and $t_1^* = 9$ in (3.52), $\delta a\left(t_1^*\right) = 41$ and $a_2 = 44$, to which corresponds by (3.62) $f_{2,2} = 8487$. The following positive and negative branches of other values of $f_{2,\left(4k-\frac{3\pm1}{2}\right)}$ for $f_1 = 21$ appear respectively for $a_{4k-2} = 211k + 44$ for all positive integers k, i.e., $a_{4k-2} = 255$, 466, 677, ..., and for $a_{4k-1} = 211k - 43 = 168$, 379, 590, ..., for which all $GM_{a,5}$ are again composites. All the corresponding values of f_2 are taken as new values of f_1 and the process is repeated for these values of f_1. The next value of a_1 for which $f_2 = 0$ and f_1 (3.62) would give a prime factor c_1 is $a_1 = 6$, yielding $f_1 = 465$ and $c_1 = 4651$ (not represented in Table 6), corresponding to $GM_{6,5} = 4651$ which is of course prime. The following values of $f_{2,\left(4k-\frac{3\pm3}{2}\right)}$ on the positive and negative branches for $f_1 = 465$ can be calculated as above, yielding new values of f_2 that

Table 6: Evolution of Values of $GM_{a,5}$, f_1, c_1 and f_2 for n = 5 and $2 \leq a \leq 50$

			f_1 =	**1**	**3**	**4**	**6**	**7**	**10**
			c_1 =	**11**	**31**	**41**	**61**	**71**	**101**
a	$GM_{a,5}$	Q_5	f_1, f_2	$f_{2\pm}$	$f_{2\pm}$	$f_{2\pm}$	$f_{2\pm}$	$f_{2\pm}$	$f_{2\pm}$
2	**31**	3	3, 0	-	0_+	-	-	-	-
3	**211**	21	21, 0	-	-	-	-	-	-
4	781	78	1, 7	7_+	-	-	-	1_+	-
5	2101	210	1, 19	19_+	-	-	-	-	-
6	**4651**	465	465, 0	-	-	-	-	-	-
7	9031	903	1, 82	82_-	-	-	-	-	-
8	15961	1596	1, 145	145_-	-	-	-	-	-
9	26281	2628	4, 64	-	-	64_+	-	-	-
10	40951	4095	3, 132	-	132_+	-	-	-	-
11	**61051**	6105	6105, 0	-	-	-	-	-	-
12	87781	8778	4, 214	-	-	214_+	-	-	-
13	122461	12246	15, 81	-	-	-	-	-	-
14	166531	16653	24, 69	-	-	-	-	-	-
15	221551	22155	1, 2014	2014_+	-	-	-	-	-
16	289201	28920	1, 2629	2629_+	-	-	474_+	-	-
17	**371281**	37128	37128, 0	-	-	-	-	-	-
18	469711	46971	1, 4270	4270_-	-	-	-	-	-
19	586531	58653	1, 5332	5332_-	-	-	-	826_+	-
20	**723901**	72390	72390, 0	-	-	-	-	-	-
21	884101	88410	33, 267	-	-	-	-	-	-
22	1069531	106953	3, 3450	-	3450_-	-	-	-	-
23	1282711	128271	54, 237	-	-	-	-	-	-
24	1526281	152628	6, 2502	-	-	-	2502_+	-	-
25	**1803001**	180300	180300, 0	-	-	-	-	-	-
26	2115751	211575	1, 19234	19234_+	-	-	-	-	-
27	2467531	246753	1, 22432	22432_+	-	-	-	-	2443_+
28	**2861461**	286146	286146, 0	-	-	-	-	-	-
29	3300781	330078	1, 3007	3007_-	-	-	-	-	3268_+
30	3788851	378885	1, 3444	3444_-	12222_-	9241_-	-	-	-
31	**4329151**	432915	432915, 0	-	-	-	-	-	-
32	**4925281**	492528	492528, 0	-	-	-	-	-	-
33	5580961	558096	3, 18003	-	18003_+	13612_-	-	-	-
34	6300031	630003	88, 715	-	-	-	-	-	-
35	**7086451**	708645	708645, 0	-	-	-	-	-	-
36	**7944301**	794430	794430, 0	-	-	-	-	-	-
37	8877781	887778	1, 80707	80707_+	-	-	-	-	-
38	9891211	989121	1, 89920	89920_+	-	-	16215_-	-	-
39	10989031	1098903	25, 4378	-	-	-	-	-	-
40	12175801	1217580	1, 110689	110689_-	-	-	-	-	-
41	13456201	1345620	1, 122329	122329_-	43407_+	-	-	-	-
42	**14835031**	1483503	1483503, 0	-	-	-	-	-	-

Table 6 (Cont.)									
			$f_1=$	**1**	**3**	**4**	**6**	**7**	**10**
			$c_1=$	**11**	**31**	**41**	**61**	**71**	**101**
a	$GM_{a,5}$	Q_5	f_1, f_2	$f_{2\pm}$	$f_{2\pm}$	$f_{2\pm}$	$f_{2\pm}$	$f_{2\pm}$	$f_{2\pm}$
43	*16317211*	1631721	15, 10806	-	-	-	-	-	-
44	*17907781*	1790778	21, 8487	-	-	-	-	-	-
45	**19611901**	1961190	1961190, 0	-	-	-	-	-	-
46	*21434851*	2143485	6, 35139	-	-	-	35139_-	-	-
47	**23382031**	2338203	2338203, 0	-	-	-	-	-	-
48	*25458961*	2545896	1, 231445	231445_+	-	-	-	-	-
49	*27671281*	2767128	1, 251557	251557_+	-	-	-	-	-
50	*30024751*	3002475	4, 73231	-	-	73231_+	-	-	-
Notes: $GM_{a,5}$ in bold (respectively italic) characters are prime (respectively composite), $Q_5 = \Delta(a-1)(2\Delta(a-1)+1)$, subscripts + and – refer to f_{2+} and f_{2-}.									

are taken as new values of f_1 to restart the process. Instead, let's calculate for the third value of $a_1 = 4$. One has from (3.62) $f_1 = 78$ to which corresponds $c_1 = 781$, which is not prime and decomposes in prime factors as 11.71, which are the prime factors of $GM_{4,5}$. The corresponding coefficients f_i are $f_1 = 1$ and $f_{2,1} = 7$. The following values of $f_{2,\left(4k-\frac{3\pm3}{2}\right)}$ on the positive and negative branches for $f_1 = 1$ appear respectively for $a_{4k-3} = 4 + kc_1 = 11k + 4$ for all positive integers k, i.e., $a_{4k-3} = 15$ (for $f_2 = 2014$), 26,37, ..., and for $a_{4k} = -(4-1-kc_1) = 11k-3$, i.e., $a_{4k} = 8$ (for $f_2 = 145$), 19, 30, ..., for which all $GM_{a,5}$ are also composites. Still for $f_1 = 1$, the second solution is $a_2 = 5$, to which corresponds $f_{2,2} = 19$ by (3.62). The following positive and negative branches of other values of $f_{2,\left(4k-\frac{3\pm1}{2}\right)}$ for $f_1 = 1$ appear respectively for $a_{4k-2} = 11k + 5$ for all positive integers k, i.e., $a_{4k-2} = 16$ (for $f_2 = 2629$), 27, 38, ..., and for $a_{4k-1} = 11k-4 = 7$ (for $f_2 = 82$), 18, 29, ..., for which all $GM_{a,5}$ are again composites. And so on.

For the simpler step by step algorithm to determine whether a $GM_{a,5}$ is prime or composite and to determine its factors, one can simply tests the ratio from (3.31)

$$R = \frac{\Delta(a-1)\left(2\Delta(a-1)+1\right)-i}{10i+1} \qquad \text{...(3.63)}$$

for all positive integers i until R is smaller than 1. Following Table 6 like above, if for a value of i, R becomes an integer, the corresponding $GM_{a,5}$ is composite.

If for all values of i until $R < 1$, R never becomes integer, the corresponding $GM_{a,5}$ is prime.

3.3. Bases a Yielding Generalized Mersenne Composites for n = 7

3.3.1. Algebraic Method

For $n = 7$, (2.5) and (1.4), with

$$Q_7(a) = \Delta(a-1)\,(2\Delta(a-1)+1)^2 \qquad \text{...(3.64)}$$

yield

$$a^6 - 3a^5 + 5a^4 - 5a^3 + 3a^2 - a - 2\,(f_1 + 14f_1f_2 + f_2) = 0 \qquad \text{...(3.65)}$$

which has two real general solutions

$$a = \frac{1 \pm \sqrt{1 + \frac{4}{3}\left(h^{\frac{1}{3}} + h^{-\frac{1}{3}} - 2\right)}}{2} \qquad \text{...(3.66)}$$

with

$$h = 27\left(f_1 + 14f_1f_2 + f_2\right) + 1 + \sqrt{27\left(f_1 + 14f_1f_2 + f_2\right)\left(27\left(f_1 + 14f_1f_2 + f_2\right) + 2\right)} \quad ...(3.67)$$

and four other complex solutions of no interest here. One of the real solutions will be a positive integer solution for the positive sign in front of the square root sign in (3.66), and if the discriminant of (3.66) is the square of an odd integer $(2K + 1)^2 = 1 + 8\Delta(K)$, i.e., if

$$f_1 + 14f_1f_2 + f_2 = \Delta(K)(2\Delta(K) + 1)^2 = \Phi_7(\Delta(K)) \quad ...(3.68)$$

as some simple algebra would show, with K positive integers, which corresponds to relation (2.7)

$$Q_7(a) = \Delta(a-1)\,(2\Delta(a-1) + 1)^2 = \Phi_7(\Delta(K)) = \Delta(K)(2\Delta(K) + 1)^2 \quad ...(3.69)$$

that gives the first integer solution in a of the form (2.8), $a = K + 1$. For integers $K = 1$ to 5, $\Delta(K)(2\Delta(K) + 1)^2$ takes the values 9, 147, 1014, 4410, 14415, etc. Only for these values does a in (3.66) take integer values greater than 1, which are further determined by the general method.

3.3.2. Excluded f_i Values

Excluded integer values are $f_i \neq 1, 4, 6, 7, 10, 11, 12, \ldots$ that do not yield integer solutions for a in (3.65). Relation (2.10) gives the three general relations of excluded f_i values for integers k,

$$(\alpha,\ \beta,\ \gamma) = (0,\ -1,\ -1) : f_i \neq -k\left(\mathrm{mod}\left(14k - 1\right)\right) \text{ for } k > 0 \quad ...(3.70)$$

$$(\alpha,\ \beta,\ \gamma) = (1,\ \pm 5,\ \pm 3) : f_i \neq \left(\pm 5k + 1\right)\left(\mathrm{mod}\left(14k \pm 3\right)\right) \text{ for } k \geq 0 \quad ...(3.71)$$

$$(\alpha,\ \beta,\ \gamma) = (1,\ \pm 3,\ \pm 5) : f_i \neq \left(\pm 3k + 1\right)\left(\mathrm{mod}\left(14k \pm 5\right)\right) \text{ for } k \geq 0 \quad ...(3.72)$$

For all other positive and negative integer values of α, β, γ complying with (2.11), the general expressions of f_i will not be different. With (2.2), forbidden forms of factors c_i (2.20) corresponding to excluded values f_i (2.59) are of the form, with integers k,

$$c_i \neq 0 \ (\mathrm{mod}\ (14k - 1)) \text{ for } k > 0 \quad ...(3.73)$$

$$c_i \neq 0 \ (\mathrm{mod}\ (14k \pm t)) \text{ for } k \geq 0 \quad ...(3.74)$$

with t being respectively 3 and 5. These forbidden forms of factors c_i are always composites and the product of at least two factors, which are multiple of integers of the form $(14j - 1)$ and/or $(14j \pm t)$, with j integers and with at least once $j = k$.

3.3.3. General Method

From (2.70), the first pair of solutions (a_{6k-5}, a_{6k}) in all k groups is

$$a_{6k-5} = a_1 + (k-1)c_1 < a_{6k} = kc_1 - a_1 + 1 \quad ...(3.75)$$

From (2.46), with the coefficients (2.26)

$$A_0 = 4\Delta(a_1 - 1)\,(3\Delta(a_1 - 1) + 2) + 1 = (3a_1(a_1 - 1) + 1)(a_1(a_1 - 1) + 1) \quad ...(3.76)$$

$$A_1 = 2(3\Delta(a_1 - 1) + 1) = 3a_1(a_1 - 1) + 2 \quad ...(3.77)$$

the function

$$F_{(2)}\left(a_1, t^* c_1\right) = \frac{-\left(3a_1\left(a_1 - 1\right) + 2\right) + \sqrt{4t^* c_1 - 3\left(a\left(a_1 - 1\right)\right)^2 - 4a_1\left(a_1 - 1\right)}}{2} \quad ...(3.78)$$

yields

$$\delta a\left(t^*\right) = \frac{-\left(2a_1 - 1\right) + \sqrt{-\left(2a_1\left(a_1 - 1\right) + 3\right) + 2\sqrt{4t^* c_1 - 3\left(a\left(a_1 - 1\right)\right)^2 - 4a_1\left(a_1 - 1\right)}}}{2} \quad ...(3.79)$$

which takes positive integer values first, for the positive sign in front of the root sign and if the first square root is greater than the first term in (3.79) yielding

$$t^* > \frac{\left(3a_1\left(a_1-1\right)+1\right)\left(a_1\left(a_1-1\right)+1\right)}{c_1} \qquad \text{...(3.80)}$$

that verifies the first condition (2.54), and second, for those values of integers t^* that yield positive integers value to $\delta a(t^*)$, i.e., if the discriminant in (3.79) is the square of an odd integer, i.e.,

$$-\left(2a_1\left(a_1-1\right)+3\right)+2\sqrt{4t^*c_1-3\left(a\left(a_1-1\right)\right)^2-4a_1\left(a_1-1\right)}=\left(2T+1\right)^2 \qquad \text{...(3.81)}$$

with T positive integers, yielding

$$t^* = \frac{\left(2\Delta(T)+1\right)\left(2\Delta(T)+2\Delta\left(a_1-1\right)+1\right)+2\Delta\left(a_1-1\right)\left(2\Delta\left(a_1-1\right)+1\right)}{c_1} \qquad \text{...(3.82)}$$

The first four smallest integer values of T such that t^* in (3.82) is integer and complies with (3.80) give t_1^* to t_4^*, yielding the four differences $\delta a\left(t_1^*\right)$ to $\delta a\left(t_4^*\right)$. These four values of $\delta a(t^*)$ are related in pairs by (2.64) and $\delta a\left(t_3^*\right)$ can be found from $\delta a\left(t_2^*\right)$ and $\delta a\left(t_4^*\right)$ from $\delta a\left(t_1^*\right)$, as

$$\delta a\left(t_3^*\right)=c_1-2a_1+1-\delta a\left(t_2^*\right)=c_1-a_1-a_3+1 \qquad \text{...(3.83)}$$

$$\delta a\left(t_4^*\right)=c_1-2a_1+1-\delta a\left(t_1^*\right)=c_1-a_1-a_2+1 \qquad \text{...(3.84)}$$

yielding, from (2.47) and (3.78),

$$t_3^*=t_2^*+\left(c_1-2a_3+1\right)\left(c_1\left(c_1-2a_3+1\right)+2a_3\left(a_3-1\right)+a_1\left(a_1-1\right)+2\right) \qquad \text{...(3.85)}$$

$$t_4^*=t_1^*+\left(c_1-2a_2+1\right)\left(c_1\left(c_1-2a_2+1\right)+2a_2\left(a_2-1\right)+a_1\left(a_1-1\right)+2\right) \qquad \text{...(3.86)}$$

Note that $\delta a\left(t_4^*\right)$ in (3.84) verifies the condition (2.57) as $a_1 < a_2$. The fourth condition (2.62) yields that

$$t_5^* > \left(c_1+2a_1-1\right)\left(c_1\left(c_1+2a_1-1\right)+3a_1\left(a_1-1\right)+2\right)+\frac{a_1\left(a_1-1\right)\left(3a_1\left(a_1-1\right)+4\right)+1}{c_1} \qquad \text{...(3.87)}$$

The first group of three pairs of solutions read then

$$\begin{aligned} &a_1 < a_2 = a_1+\delta a\left(t_1^*\right) < a_3 = a_1+\delta a\left(t_2^*\right) \\ &\quad < a_4 = a_1+\delta a\left(t_3^*\right) = c_1-a_3+1 < a_5 = a_1+\delta a\left(t_4^*\right) = c_1-a_2+1 \\ &\quad < a_6 = c_1-a_1+1 \end{aligned} \qquad \text{...(3.88)}$$

The other groups of three pairs of solutions are found by adding $(k-1)c_1$ to the first six solutions as

$$a_{6k-5} = a_1+(k-1)c_1 < \ldots < a_{6(k-1)+j} = a_j+(k-1)c_1 < \ldots < a_{6k} = a_6+(k-1)c_1 \qquad \text{...(3.89)}$$

for $1 \le j \le 6$ and for all positive integers k. These relations for a can be combined into the single general relation (2.78), yielding all values of solutions a.

$$a_{\left(6k-\frac{5\pm(2j-1)}{2}\right)} = \frac{(1-c_1)\pm\left(2a_{(4-j)}-c_1-1\right)}{2}+kc_1 \quad ...(3.90)$$

for $1 \le j \le 3$ for all positive integers k. The differences $\delta f_2(r)$, $\delta f_2(s)$ and $\delta f_2(t^*)$ are found from (2.79), (2.80) and (2.49) and a single general relation for $f_{2,\left(6k-\frac{5\pm(2j-1)}{2}\right)}$ similar to (3.18) and (3.60) can be found in function of a_1, c_1 and k. However, for increasing n, these expressions become more and more complicated and values of f_2 increase very rapidly.

3.3.4. Example of Calculations

From the algebraic method, starting with the first non-excluded value of $f_1 = 2$, yielding $c_1 = 29$, the smallest value of f_2 that satisfies (3.68) is $f_{2,1} = 5$ to which corresponds $K = 2$ as

$$c_1 f_{2,1} + f_1 = 29 \cdot 5 + 2 = \Delta(K=2)(2\Delta(K=2)+1)^2 = 147 \quad ...(3.91)$$

yielding from (2.8) the first seed solution $a_1 = 3$ for this value of $f_1 = 2$. From (3.82), the first two integer values of T are 4 and 15, and yield the first integer values of $t_1^* = 21$ and $t_2^* = 41$, giving from (3.79), $\delta a\left(t_1^*\right) = 2$ and $\delta a\left(t_2^*\right) = 3$, yielding $a_2 = 5$ and $a_3 = 6$. The corresponding values of f_2 are 152 and 497. From (3.88), the other solutions for $k = 1$, are $a_4 = 24$, $a_5 = 25$ and $a_6 = 27$. Other values are given in Table 7 for $1 \le f_1 \le 10$ and $k = 1$ to 3, with the first three values of f_2. Corresponding composite $GM_{a,7}$ can be found in Sequence A022523 in (Sloane, 2024).

Note as well that for $f_1 = 9$, which is the value of $\Phi_7(\Delta(K))$ for $K = 1, f_2 = 0$ in (3.68), meaning that $a_1 = 2$ does not yield a composite but a prime generalized Mersenne, which is the Mersenne prime $M_7 = 127$.

Like for the previous cases, one can also test the ratio

$$R = \frac{\Delta(a-1)\left(2\Delta(a-1)+1\right)^2 - i}{14i+1} \quad ...(3.92)$$

to find whether a $GM_{a,7}$ is prime or composite.

3.4. Bases a Yielding Generalized Mersenne Composites for n = 11

3.4.1. Algebraic Method

Similarly, for the next prime value of the exponent $n = 11$, from (2.5) and (1.4), with

$$Q_{11}(a) = \Delta(a-1)\,(2\Delta(a-1)+1)\,(2\Delta(a-1)\,(2\Delta(a-1)+1)\,(2\Delta(a-1)+3)+1) \quad ...(3.93)$$

one has

$$a^{10} - 5a^9 + 15a^8 - 30a^7 + 42a^6 - 42a^5 + 30a^4 - 15a^3 + 5a^2 - a - 2(f_1 + 22f_1f_2 + f_2) = 0 \quad ...(3.94)$$

which has at least two real solutions in a, whose integer values can be found in the form (2.5), if

$$f_1 + 22f_1f_2 + f_2 = \Phi_{11}(\Delta(K)) = \Delta(K)\,(2\Delta(K)+1)\,(2\Delta(K)\,(2\Delta(K)+1)\,(2\Delta(K)+3)+1) \quad ...(3.95)$$

with K positive integers, which corresponds to relation (3.93), giving the first integer solution in a of the form (2.8), $a = K + 1$. For integers $K = 1$ to 3, $\Phi_{11}(\Delta(K))$ takes the values 93, 7959, 182598, ... Only for these values does a in (3.94) take integer values greater than 1, which are further determined by the general method.

3.4.2. Excluded f_i Values

Excluded integer values are $f_i \ne 2, 5, 6, 7, 8, 10, \ldots$ that do not yield integer solutions for a. Like above, (2.10) gives the five general relations of excluded f_i values for all non-negative integers k for (α, β, γ) taking values:

$$(0,-1,-1)\, f_i \ne -k(\mathrm{mod}\,(22k-1)) \quad ...(3.96)$$

$$(-1,7,\pm3)\, f_i \ne (7k-1)\,(\mathrm{mod}\,(22k\pm3)) \quad ...(3.97)$$

$$(2,\pm9,\pm5)\, f_i \ne (\pm9k+2)\,(\mathrm{mod}\,(22k\pm5)) \quad ...(3.98)$$

Table 7: Values of a_κ and $f_{2,\kappa}$ Yielding Composite $GM_{a,7}$ for n = 7

		k = 1		k = 2		k = 3	
f_1	c_1	a_κ	$f_{2,\kappa}$	a_κ	$f_{2,\kappa}$	a_κ	$f_{2,\kappa}$
2	29	3	5	32	16864838	61	845770532
		5	152	34	24396323	63	1027995155
		6	497	35	29103317	64	1130704118
		24	2910458	53	361180880	82	5053582763
		25	3736562	54	404468153	83	5437201805
		27	5981633	56	504087722	85	6277522592
3	43	14	70872	57	378409707	100	11284825929
		15	108714	58	420409758	101	11982616629
		20	641409	63	693299058	106	16034856534
		24	1962867	67	1005898512	110	20046175944
		29	6240777	72	1553974827	115	26204499864
		30	7674624	73	1689012483	116	27607949169
5	71	3	2	74	1110545024	145	64112798885
		16	98165	87	2950399664	158	107502085202
		22	697454	93	4411898327	164	134535998039
		50	103648820	121	21561120305	192	347327466509
		56	205894985	127	28859495324	198	417953698502
		69	727730819	140	51902084075	211	612683956250
8	113	15	41369	128	19010172728	241	856231952624
		40	16819844	153	55658816228	266	1549838391149
		46	39284579	159	70159868897	272	1772221206512
		68	418632878	181	153028196714	294	2828436912167
		74	697776077	187	186200218007	300	3193585687034
		99	4041707627	212	396064508582	325	5166363067634
9	127	2	0	129	17726293065	256	1095267058953
		18	113544	145	35842588353	272	1576858238865
		42	20126580	169	90112457685	296	2621304129213
		86	1538286498	213	362519950119	340	6028505548923
		110	6787287918	237	688910587995	364	9082321748295
		126	15383781558	253	1020335724819	380	11760904789263

Note: Where $\kappa = (6k - i)$ with $5 \geq i \geq 0$ respectively from the first to the sixth line for each value of f_1. Note a_1 yields a prime $GM_{a,7}$ for $f_1 = 9$ and $k = 1$.

$$(-1, 3, \pm 7) f_i \neq (3k - 1) \pmod{(22k \pm 7)} \qquad \text{...(3.99)}$$

$$(2, \pm 5, \pm 9) f_i \neq (\pm 5k + 2) \pmod{(22k \pm 9)} \qquad \text{...(3.100)}$$

with $k > 0$ for (3.96) and $k \geq 0$ for (3.97) to (3.100). For all other positive and negative integer values of α, β, and γ complying with (2.11), the general expressions of f_i will not be different. With (2.2), forbidden forms of factors c_i (2.20) corresponding to excluded values f_i (3.96) to (3.100) are of the form

$$c_i \neq 0 \pmod{(22k - 1)} \text{ for } k > 0 \qquad \text{...(3.101)}$$

$$c_i \neq 0 \pmod{(22k \pm t)} \text{ for } k \geq 0 \qquad \text{...(3.102)}$$

with t being respectively 3, 5, 7 and 9. These forbidden forms of factors c_i are always composites and the product of at least two factors, which are multiple of integers of the form $(22j-1)$ and/or $(22k \pm t)$, with j integers and with at least once $j = k$.

3.4.3. General Method

From (2.70), the first pair of solutions (a_{10k-9}, a_{10k}) in all k groups is

$$a_{10k-9} = a_1 + (k-1)c_1 < a_{10k} = kc_1 - a_1 + 1 \quad \text{...(3.103)}$$

From (2.46), with the coefficients (2.26),

$$A_0 = 80(\Delta(a_1-1))^3 (\Delta(a_1-1)+2) + 4\Delta(a_1-1)(21\Delta(a_1-1)+4)+1 \quad \text{...(3.104)}$$

$$A_1 = 2[20(\Delta(a_1-1))^2 (2\Delta(a_1-1)+3) + 21\Delta(a_1-1)+4] \quad \text{...(3.105)}$$

$$A_2 = 40\Delta(a_1-1)(\Delta(a_1-1)+1)+7 \quad \text{...(3.106)}$$

$$A_3 = 5(2\Delta(a_1-1)+1) \quad \text{...(3.107)}$$

one has the function

$$F_{(4)}\left(a_1, t^*c_1\right) = \frac{-2H + \sqrt{H - 2\left(G - \sqrt{G^2 - 4\left(A_0 - t^*c_1\right)}\right)}}{2} \quad \text{...(3.108)}$$

where the sign in front of the first (respectively the second) square root must be positive (respectively negative) for $F_{(4)}(a_1, t^*c_1)$ to be positive, and with

$$H = \frac{A_3 \pm \sqrt{A_3^2 + 4\left(G - A_2\right)}}{4} \quad \text{...(3.109)}$$

$$G = \sqrt[3]{U + \sqrt{U^2 + V^3}} + \sqrt[3]{U - \sqrt{U^2 + V^3}} + \frac{A_2}{3} \quad \text{...(3.110)}$$

$$U = \frac{A_2\left(4\left(A_0 - t^*c_1\right) - A_1A_3\right)}{6} + \frac{\left(A_3^2 + 4A_2\right)\left(A_0 - t^*c_1\right) + A_1^2}{2} + \left(\frac{A_2}{3}\right)^3 \quad \text{...(3.111)}$$

$$V = \frac{A_1A_3 - 4\left(A_0 - t^*c_1\right)}{3} - \left(\frac{A_2}{3}\right)^2 \quad \text{...(3.112)}$$

The sign in front of the square root in (3.109) must be positive or negative depending on the value of t^*. Relation (3.108) yields

$$\delta a\left(t^*\right) = \frac{1}{2}\left(-\left(2a_1 - 1\right) + \left[3\left(3a_1\left(a_1 - 1\right) + 2\right) \pm \sqrt{A_3^2 + 4\left(G - A_2\right)} \mp 2\sqrt{H - 2\left(G \pm \sqrt{G^2 - 4\left(A_0 - t^*c_1\right)}\right)}\right]^{\frac{1}{2}}\right) \quad \text{...(3.113)}$$

which takes positive integer values first, for the positive sign in front of the first root sign and if the first square root is greater than the first term in (3.113) and second, for those values of integers t^* that yield positive integers value to $\delta a(t^*)$, i.e., if the discriminant in (3.113) is the square of an odd integer $(2T+1)^2$

$$3\left(3a_1\left(a_1 - 1\right) + 2\right) \pm \sqrt{A_3^2 + 4\left(G - A_2\right)} \mp 2\sqrt{H - 2\left(G \pm \sqrt{G^2 - 4\left(A_0 - t^*c_1\right)}\right)} = \left(2T + 1\right)^2 \quad \text{...(3.114)}$$

with T positive integers. The first condition (2.54) is verified if

$$t^* > \frac{80\left(\Delta\left(a_1 - 1\right)\right)^3\left(\Delta\left(a_1 - 1\right) + 2\right) + 4\Delta\left(a_1 - 1\right)\left(21\Delta\left(a_1 - 1\right) + 8\right) + 1}{c_1} \quad \text{...(3.115)}$$

The third condition (2.59) is verified if

$$t_8^* < c_1^3\left(c_1 - 2a_1 + 1\right)^4 + A_3 c_1^2\left(c_1 - 2a_1 + 1\right)^3 + A_2 c_1\left(c_1 - 2a_1 + 1\right)^2 + A_1\left(c_1 - 2a_1 + 1\right) + \frac{A_0}{c_1} \quad ...(3.116)$$

The fourth condition (2.62) is verified if

$$t_9^* > c_1^3\left(c_1 - 2a_1 + 1\right)^4 + A_3 c_1^2\left(c_1 - 2a_1 + 1\right)^3 + A_2 c_1\left(c_1 - 2a_1 + 1\right)^2 + A_1\left(c_1 - 2a_1 + 1\right) + \frac{A_0}{c_1} \quad ...(3.117)$$

The first eight smallest integer values of T in (3.114) yield t_1^* to t_8^* that satisfy (3.115) and (3.116) and that render $\delta a\left(t_1^*\right)$ to $\delta a\left(t_8^*\right)$ (3.113) integers. These eight values of $\delta a(t^*)$ are related in pairs by (2.64) and $\delta a\left(t_{9-j}^*\right)$ can be found from $\delta a\left(t_j^*\right)$, as

$$\delta a\left(t_{9-j}^*\right) = c_1 - 2a_1 + 1 + \delta a\left(t_j^*\right) = c_1 - a_1 - a_{j+1} + 1 \quad ...(3.118)$$

for $1 \le j \le 4$. Note that $\delta a\left(t_8^*\right)$ in (3.118) verifies again the condition (2.57) as $a_1 < a_2$. The first group of five pairs of solutions read then

$$\begin{aligned} a_1 < a_2 &= a_1 + \delta a\left(t_1^*\right) < a_3 = a_1 + \delta a\left(t_2^*\right) < a_4 = a_1 + \delta a\left(t_3^*\right) \\ &< a_5 = a_1 + \delta a\left(t_4^*\right) < a_6 = a_1 + \delta a\left(t_5^*\right) = c_1 - a_5 + 1 \\ &< a_7 = a_1 + \delta a\left(t_6^*\right) = c_1 - a_4 + 1 < a_8 = a_1 + \delta a\left(t_7^*\right) = c_1 - a_3 + 1 \\ &< a_9 = a_1 + \delta a\left(t_8^*\right) = c_1 - a_2 + 1 < a_{10} = c_1 - a_1 + 1 \end{aligned} \quad ...(3.119)$$

The other groups of five pairs of solutions are found by adding $(k-1)c_1$ to the first ten solutions as

$$a_{10k-9} = a_1 + (k-1)c_1 < \ldots < a_{10(k-1)+j} = a_j + (k-1)c_1 < \ldots < a_{10k} = a_{10} + (k-1)c_1 \quad ...(3.120)$$

for $1 \le j \le 10$ and for all positive integers k. The relations for a in (3.119) can be combined into the single general relation (2.78), yielding all values of solutions a

$$a_{\left(10k - \frac{9 \pm (2j-1)}{2}\right)} = \frac{\left(1 - c_1\right) \pm \left(2a_{(6-j)} - c_1 - 1\right)}{2} + kc_1 \quad ...(3.121)$$

with $1 \le j \le 5$ and $k > 0$.

3.4.4. Example of Calculations

From the algebraic method, starting with the first non-excluded value of $f_1 = 1$, yielding $c_1 = 23$, the smallest value of f_2 that satisfies (3.95) is $f_{2,1} = 4$ to which corresponds $K = 1$ as

$$c_1 f_{2,1} + f_1 = 23 \cdot 4 + 1 = \Delta(1)\,(2\Delta(1)+1)\,(2\Delta(1)\,(2\Delta(1)+1)\,(2\Delta(1)+3)+1) = 93 \quad ...(3.122)$$

yielding from (2.8) the first seed solution $a_1 = 2$ for this value of $f_1 = 1$. From (3.113), the values of t^* for t_1^* to t_8^* vary between 240 and 1375867180, yielding respectively the first four $\delta a(t^*)$, i.e., $\delta a\left(t_1^*\right) = 1$, $\delta a\left(t_2^*\right) = 2$, $\delta a\left(t_3^*\right) = 7$, and $\delta a\left(t_4^*\right) = 9$, yielding $a_2 = 3$, $a_3 = 4$, $a_4 = 9$, and $a_5 = 11$. The corresponding values of f_2 vary between 346 and 366228598. From (3.119), the other solutions for $k = 1$, are $a_6 = 13$, $a_7 = 15$, $a_8 = 20$, $a_9 = 21$ and $a_{10} = 22$. Other values are given in Table 8 for $1 \le f_1 \le 10$ and $k = 1$ to 3; the corresponding values of f_2 can be found in (Pletser, 2024b). Corresponding composite $GM_{a,11}$ can be found in Sequence A022527 in (Sloane, 2024).

Table 8: Values of a_κ Yielding Composite $GM_{a,11}$ for $n = 11$

$k =$	$f_1 = 1, c_1 = 23$			$f_1 = 3, c_1 = 67$			$f_1 = 4, c_1 = 89$			$f_1 = 9, c_1 = 199$		
	1	2	3	1	2	3	1	2	3	1	2	3
$a_{\kappa+}$	2	25	48	12	79	146	2	91	180	62	261	460
	3	26	49	15	82	149	7	96	185	63	262	461
	4	27	50	17	84	151	24	113	202	69	268	467
	9	32	55	25	92	159	31	120	209	81	280	479
	11	34	57	32	99	166	38	127	216	83	282	481
$a_{\kappa-}$	13	36	59	36	103	170	52	141	230	117	316	515
	15	38	61	43	110	177	59	148	237	119	318	517
	20	43	66	51	118	185	66	155	244	131	330	529
	21	44	67	53	120	187	83	172	261	137	336	535
	22	45	68	56	123	190	88	177	266	138	337	536

Note: Where $\kappa_+ = (10k - i)$ with $9 \geq i \geq 5$ and $\kappa_- = (10k - j)$ with $4 \geq j \geq 0$, respectively from the first to the fifth row for $a_{\kappa+}$ and $a_{\kappa-}$.

Note that the first value of a for $f_1 = 1$ and $k = 1$, i.e., $a_1 = 2$, does not yield a prime as $f_2 = 4$ is not nil, which explains why the fifth Mersenne number M_{11} is not a prime but a composite, $GM_{2,11} = M_{11} = 2047 = 23 \cdot 89 = (2 \cdot 11 \cdot 1 + 1)\,(2 \cdot 11 \cdot 4 + 1)$.

Like for the previous cases, one can also test the ratio

$$R = \frac{\Delta\left(2\Delta+1\right)\left(2\Delta\left(2\Delta+1\right)\left(2\Delta+3\right)+1\right)-i}{22i+1} \quad ...(3.123)$$

with $\Delta = \Delta(a - 1)$, to find whether a $GM_{a,11}$ is prime or composite.

3.5. Bases a Yielding Generalized Mersenne Composites for n = 13

3.5.1. Algebraic Method

For $n = 13$, from (2.5) and (1.4), with

$$Q_{13}(a) = \Delta(a-1)\,(2\Delta(a-1)+1)^2\,(2\Delta(a-1)\,(2\Delta(a-1)\,(2\Delta(a-1)+5)+3)+1) \quad ...(3.124)$$

one has

$$a^{12} - 6a^{11} + 22a^{10} - 55a^9 + 99a^8 - 132a^7 + 132a^6 - 99a^5 + 55a^4 - 22a^3 + 6a^2 - a - 2(f_1 + 26f_1f_2 + f_2) = 0 \quad ...(3.125)$$

which has at least two real solutions in a, whose integer values can be found in the form (2.5), if

$$f_1 + 26f_1f_2 + f_2 = \Phi_{13}(\Delta(K)) = \Delta(K)\,(2\Delta(K)+1)^2\,[2\Delta(K)\,(2\Delta(K)\,(2\Delta(K)+5)+3)+1] \quad ...(3.126)$$

with K positive integers, which corresponds to (3.124), giving the first integer solution in a of the form (2.8), $a = K + 1$. For integers $K = 1$ to 3, $\Phi_{13}(\Delta(K))$ takes the values 315, 61005, 2519790, ... Only for these values does a in (3.125) take integer values greater than 1, which are further determined by the general method.

3.5.2. Excluded f_i Values

Excluded integer values are $f_i \neq 1, 4, 7, 8, 9, 10, 11, ...$ that do not yield integer solutions for a. As above, (2.10) gives the five general relations of excluded f_i values for all integers k for (α, β, γ) taking values:

$$(0, -1, -1)\, f_i \neq -k \pmod{26k-1} \quad ...(3.127)$$

$$(-1, \pm 9, \pm 3)\, f_i \neq (\pm 9k+1) \pmod{26k \pm 3} \quad ...(3.128)$$

$$(-1, \mp 5, \pm 5)\, f_i \neq (\mp 5k-1) \pmod{26k \pm 5} \quad ...(3.129)$$

$$(-3, \mp 11, \pm 7)\, f_i \neq (\mp 11k-3) \pmod{26k \pm 7} \quad ...(3.130)$$

$$(1, \pm 3, \pm 9) f_i \neq (\pm 3k + 1) \pmod{(26k \pm 9)} \qquad ...(3.131)$$

$$(-3, \pm 7, \mp 11) f_i \neq (\pm 7k - 3) \pmod{(26k \mp 11)} \qquad ...(3.132)$$

with $k > 0$ for (3.127) and $k \geq 0$ for (3.128) to (3.132). For all other positive and negative integers α, β and γ complying with (2.11), the general expressions of f_i will not be different. With (2.2), forbidden forms of factors c_i (2.20) corresponding to excluded values f_i (3.127) to (3.132) are of the form

$$c_i \neq 0 \pmod{(26k - 1)} \text{ for } k > 0 \qquad ...(3.133)$$

$$c_i \neq 0 \pmod{(26k \pm t)} \text{ for } k \geq 0 \qquad ...(3.134)$$

with t being all odd integers from 3 to 11. These forbidden forms of factors c_i are always composites and the product of at least two factors, which are multiple of integers of the form $(26j - 1)$ and/or $((26j \pm t)$, with j integers and with at least once $j = k$.

3.5.3. General Method

From (2.70), the first pair of solutions (a_{12k-11}, a_{12k}) in all k groups is

$$a_{12k-11} = a_1 + (k-1)c_1 < a_{12k} = kc_1 - a_1 + 1 \qquad ...(3.135)$$

From (2.46), with the coefficients (2.26)

$$A_0 = 4\Delta(4\Delta(\Delta(12\Delta^2 + 7(5\Delta + 4)) + 9) + 5) + 1 \qquad ...(3.136)$$

$$A_1 = 8\Delta(2\Delta(5\Delta(3\Delta + 7) + 21) + 9) + 5 \qquad ...(3.137)$$

$$A_2 = 4(2\Delta(5\Delta(4\Delta + 7) + 14) + 3) \qquad ...(3.138)$$

$$A_3 = 2(5\Delta(6\Delta + 7) + 7) \qquad ...(3.139)$$

$$A_4 = 12\Delta + 7 \qquad ...(3.140)$$

where, for convenience, Δ is written instead of $\Delta(a_1 - 1)$, let $F_{(5)}(a_1, t^* c_1)$ be one of the real roots of the polynomial (2.46) with t^* the integer t that yields integer values to δa

$$\delta a\left(t^*\right) = \frac{-(2a_1 - 1) + \sqrt{(2a_1 - 1)^2 + 4F_{(5)}\left(a_1, t^* c_1\right)}}{2} \qquad ...(3.141)$$

The first condition (2.54) is verified if

$$t^* > \frac{4\Delta\left(4\Delta\left(\Delta\left(12\Delta^2 + 7(5\Delta + 4)\right) + 9\right) + 5\right) + 1}{c_1} \qquad ...(3.142)$$

The third condition (2.59) is verified if

$$t_{10}^* < c_1^4\left(c_1 - 2a_1 + 1\right)^5 + A_4 c_1^3\left(c_1 - 2a_1 + 1\right)^4 + A_3 c_1^2\left(c_1 - 2a_1 + 1\right)^3 + A_2 c_1\left(c_1 - 2a_1 + 1\right)^2 + A_1\left(c_1 - 2a_1 + 1\right) + \frac{A_0}{c_1} \qquad ...(3.143)$$

The fourth condition (2.62) is verified if

$$t_{11}^* > c_1^4\left(c_1 + 2a_1 - 1\right)^5 + A_4 c_1^3\left(c_1 + 2a_1 - 1\right)^4 + A_3 c_1^2\left(c_1 + 2a_1 - 1\right)^3 + A_2 c_1\left(c_1 + 2a_1 - 1\right)^2 + A_1\left(c_1 + 2a_1 - 1\right) + \frac{A_0}{c_1} \qquad ...(3.144)$$

The first ten integer values of t_1^* to t_{10}^* that satisfy (3.142) and (3.143) and that render $\delta a\left(t_1^*\right)$ to $\delta a\left(t_{10}^*\right)$ integers, are related in pairs by (2.64) and can be found from $\delta a\left(t_j^*\right)$, as

$$\delta a\left(t^*_{11-j}\right) = c_1 - 2a_1 + 1 - \delta a\left(t^*_j\right) = c_1 - a_1 - a_{j+1} + 1 \qquad ...(3.145)$$

for $1 \le j \le 5$. The first group of six pairs of solutions can now be written as

$$\begin{aligned} a_1 &< a_2 = a_1 + \delta a\left(t^*_1\right) < a_3 = a_1 + \delta a\left(t^*_2\right) < a_4 = a_1 + \delta a\left(t^*_3\right) \\ &< a_5 = a_1 + \delta a\left(t^*_4\right) < a_6 = a_1 + \delta a\left(t^*_5\right) < a_7 = a_1 + \delta a\left(t^*_6\right) = c_1 - a_6 + 1 \\ &< a_8 = a_1 + \delta a\left(t^*_7\right) = c_1 - a_5 + 1 < a_9 = a_1 + \delta a\left(t^*_8\right) = c_1 - a_4 + 1 \\ &< a_{10} = a_1 + \delta a\left(t^*_9\right) = c_1 - a_3 + 1 < a_{11} = a_1 + \delta a\left(t^*_{10}\right) = c_1 - a_2 + 1 \\ &< a_{12} = c_1 - a_1 + 1 \end{aligned} \qquad ...(3.146)$$

The other groups of six pairs of solutions are found by adding $(k-1)c_1$ to the first twelve solutions as

$$a_{12k-11} = a_1 + (k-1)c_1 < ... < a_{12(k-1)+j} = a_j + (k-1)c_1 < ... < a_{12k} = a_{12} + (k-1)c_1 \qquad ...(3.147)$$

for $1 \le j \le 12$ and for all positive integers k. The relations for a in (3.146) can be combined into the single general relation (2.78), yielding all values of solutions a

$$a_{\left(12k - \frac{11 \pm (2j-1)}{2}\right)} = \frac{(1-c_1) \pm \left(2a_{(7-j)} - c_1 - 1\right)}{2} + kc_1 \qquad ...(3.148)$$

with $1 \le j \le 6$ and $k > 0$.

3.5.4. Calculation Methods

From the algebraic method, starting with the first non-excluded value of $f_1 = 2$, yielding $c_1 = 53$, the smallest value of f_2 that satisfies (3.126) is $f_{2,1} = 1151$ to which corresponds $K = 2$ as

$$c_1 f_{2,1} + f_1 = 53 \cdot 1151 + 2 = \Delta(2)\,(2\Delta(2)+1)^2\,(2\Delta(2)\,(2\Delta(2)\,(2\Delta(2)+5)+3)+1) = 61005 \qquad ...(3.149)$$

yielding from (2.8) the first seed solution $a_1 = 3$ for this value of $f_1 = 2$. Parameters t^*_1 to t^*_{10} vary between 3379565 and 642166339773, yielding respectively the first five $\delta a(t^*)$, i.e., $\delta a\left(t^*_1\right) = 4$, $\delta a\left(t^*_2\right) = 13$, $\delta a\left(t^*_3\right) = 17$, $\delta a\left(t^*_4\right) = 19$, and $\delta a\left(t^*_5\right) = 20$, yielding $a_2 = 7$, $a_3 = 16$, $a_4 = 20$, $a_5 = 22$, and $a_6 = 23$. From (3.146), the other solutions for $k = 1$, are $a_7 = 31$, $a_8 = 32$, $a_9 = 34$, $a_{10} = 38$, $a_{11} = 47$, and $a_{12} = 51$. Other values are given in Table 9 for $2 \le f_1 \le 10$ and $k = 1$ to 3; the corresponding values of f_2 can be found in (Pletser, 2024b). Corresponding composite $GM_{a,13}$ can be found in Sequence A022529 in (Sloane, 2024).

Note that the first value of a for $f_1 = 315$ and $k = 1$, i.e., $a_1 = 2$, one has $f_2 = 0$ in (3.149), meaning that it does not yield a composite but a generalized Mersenne prime, which is the Mersenne prime $M_{13} = 8191$.

Like for the previous cases, one can also test the ratio

$$R = \frac{\Delta(2\Delta+1)^2\left(2\Delta\left(2\Delta\left(2\Delta+5\right)+3\right)+1\right) - i}{26i+1} \qquad ...(3.150)$$

with $\Delta = \Delta(a-1)$, to find whether a $GM_{a,13}$ is prime or composite.

3.6. Bases a Yielding Generalized Mersenne Composites for n = 17

3.6.1. Algebraic Method

For $n = 17$, from (2.5) and (1.4), with

Table 9: Values of a_κ Yielding Composite $GM_{a,13}$ for n = 13												
	f_1 = 1, c_1 = 53			**f_1 = 3, c_1 = 79**			**f_1 = 5, c_1 = 131**			**f_1 = 6, c_1 = 157**		
k =	**1**	**2**	**3**	**1**	**2**	**3**	**1**	**2**	**3**	**1**	**2**	**3**
$a_{\kappa+}$	3	56	109	5	84	163	4	135	266	8	165	322
	7	60	113	7	86	165	19	150	281	12	169	326
	16	69	122	15	94	173	21	152	283	22	179	336
	20	73	126	22	101	180	31	162	293	29	186	343
	22	75	128	32	111	190	59	190	321	63	220	377
	23	76	129	35	114	193	63	194	325	70	227	384
$a_{\kappa-}$	31	84	137	45	124	203	69	200	331	88	245	402
	32	85	138	48	127	206	73	204	335	95	252	409
	34	87	140	58	137	216	101	232	363	129	286	443
	38	91	144	65	144	223	111	242	373	136	293	450
	47	100	153	73	152	231	113	244	375	146	303	460
	51	104	157	75	154	233	128	259	390	150	307	464

Note: Where $\kappa_+ = (12k - i)$ with $11 \geq i \geq 6$ and $\kappa_- = (12k - j)$ with $5 \geq j \geq 0$, respectively from the first to the sixth row for $a_{\kappa+}$ and $a_{\kappa-}$.

$$Q_{17}(a) = \Delta(2\Delta+1)(2\Delta(2\Delta+1)((2\Delta)^2(2\Delta+3)(2\Delta+7)+2(14\Delta+3)+1) \quad ...(3.151)$$

with $\Delta = \Delta(a-1)$, one has

$$a^{16} - 8a^{15} + 40a^{14} - 140a^{13} + 365a^{12} - 728a^{11} + 1144a^{10} - 1452a^9 + 1534a^8$$

$$-1248a^7 + 780a^6 - 364a^5 + 140a^4 - 40a^3 + 8a^2 - a - 2(f_1 + 34f_1f_2 + f_2) = 0 \quad ...(3.152)$$

which has at least two real solutions in a, whose integer values can be found in the form (2.5), if

$$f_1 + 34f_1f_2 + f_2 = \Phi_{17}(\Delta(K)) = \Delta(2\Delta+1)(2\Delta(2\Delta+1)((2\Delta)^2(2\Delta+3)(2\Delta+7)+2(14\Delta+3))+1) \quad ...(3.153)$$

with $\Delta = \Delta(K)$, and K positive integers, which corresponds to (3.151), giving the first integer solution in a of the form (2.8), $a = K+1$. For integers $K = 1$ to 3, 17 ($\Delta(K)$) takes the values 3855, 3794385, 501492030, ... Only for these values does a in (3.152) take integer values greater than 1, which are further determined by the general method.

3.6.2. Excluded f_i Values

Excluded integer values are $f_i \neq$ 1, 2, 5, 6, 8, 10, ... that do not yield integer solutions for a. As previously, (2.10) gives the eight general relations of excluded f_i values for all integers k for (α, β, γ) taking values:

$$(0,-1,-1)\, f_i \neq -k \,(\mathrm{mod}\,(34k-1)) \quad ...(3.154)$$

$$(-1,\mp 11,\pm 3)\, f_i \neq (\mp 11k-1)\,(\mathrm{mod}\,(34k\pm 3)) \quad ...(3.155)$$

$$(1,\pm 7,\pm 5)\, f_i \neq (\pm 7k+1)\,(\mathrm{mod}\,(34k\pm 5)) \quad ...(3.156)$$

$$(1,\pm 5,\pm 7)\, f_i \neq (\pm 5k+1)\,(\mathrm{mod}\,(34k\pm 7)) \quad ...(3.157)$$

$$(5,\pm 19,\pm 9)\, f_i \neq (\pm 19k+5)\,(\mathrm{mod}\,(34k\pm 9)) \quad ...(3.158)$$

$$(-1,\mp 3,\pm 11)\, f_i \neq (\mp 3k-1)\,(\mathrm{mod}\,(34k\pm 11)) \quad ...(3.159)$$

$$(-5,\mp 13,\pm 13)\, f_i \neq (\mp 13k-5)\,(\mathrm{mod}\,(34k\pm 13)) \quad ...(3.160)$$

$$(-4,\mp 9,\pm 15)\, f_i \neq (\mp 9k-4)\,(\mathrm{mod}\,(34k\pm 15)) \quad ...(3.161)$$

with $k > 0$ for (3.154) and $k \geq 0$ for (3.155) to (3.161). For all other positive and negative integers α, β, and γ complying with (2.11), the general expressions of f_i will not be different. With (2.2), forbidden forms of factors c_i (2.20) corresponding to excluded values f_i (3.154) to (3.161) are of the form, with integers k,

$c_i \neq 0 \pmod{(34k-1)}$ for $k > 0$ ...(3.162)

$c_i \neq 0 \pmod{(34k \pm t)}$ for $k \geq 0$ ...(3.163)

with t being respectively all odd integers from 3 to 15. These forbidden forms of factors c_i are always composites and the product of at least two factors, which are multiple of integers of the form $(34j-1)$ and/or $(34j \pm t)$, with j integers and with at least once $j = k$.

3.6.3. General Method

From (2.70), the first pair of solutions (a_{16k-15}, a_{16k}) in all k groups is

$$a_{16k-15} = a_1 + (k-1)c_1 < a_{16k} = kc_1 - a_1 + 1 \quad ...(3.164)$$

From (2.46), with the coefficients (2.26),

$$A_0 = 4\Delta(2\Delta(4\Delta(4\Delta^2(8\Delta^2 + 21(2\Delta+3)) + 55(3\Delta+1)) + 39) + 7) + 1 \quad ...(3.165)$$

$$A_1 = 4\Delta(2\Delta(28\Delta^2(4\Delta(2\Delta+9) + 45) + 165(4\Delta+1)) + 39) + 7 \quad ...(3.166)$$

$$A_2 = 2(4\Delta(56\Delta^2(\Delta(4\Delta+15) + 30) + 55(6\Delta+1)) + 13) \quad ...(3.167)$$

$$A_3 = 5(4\Delta(14\Delta(2\Delta+3)^2 + 33) + 11) \quad ...(3.168)$$

$$A_4 = 2(14\Delta(4\Delta(4\Delta+9) + 3) + 33) \quad ...(3.169)$$

$$A_5 = 14(4\Delta(2\Delta+3) + 3) \quad ...(3.170)$$

$$A_6 = 4(4\Delta+3) \quad ...(3.171)$$

where, for convenience, Δ was written instead of $\Delta(a_1-1)$, let $F_{(7)}(a_1, t^* c_1)$ be one of the real roots of the polynomial (2.46) and with t^* the value of the integer t that yields integer values to

$$\delta a\left(t^*\right) = \frac{-(2a_1-1) + \sqrt{(2a_1-1)^2 + 4F_{(7)}\left(a_1, t^* c_1\right)}}{2} \quad ...(3.172)$$

The first condition (2.54) is verified if

$$t^* > \frac{4\Delta\left(2\Delta\left(4\Delta\left(4\Delta^2\left(8\Delta^2 + 21(2\Delta+3)\right) + 55(3\Delta+3)\right) + 39\right) + 7\right) + 1}{c_1} \quad ...(3.173)$$

The third condition (66) is verified if

$$t_{14}^* < c_1^6\left(c_1 - 2a_1 + 1\right)^7 + A_6 c_1^5\left(c_1 - 2a_1 + 1\right)^6 + A_5 c_1^4\left(c_1 - 2a_1 + 1\right)^5 + A_4 c_1^3\left(c_1 - 2a_1 + 1\right)^4 + A_3 c_1^2\left(c_1 - 2a_1 + 1\right)^3$$
$$+ A_2 c_1\left(c_1 - 2a_1 + 1\right)^2 + A_1\left(c_1 - 2a_1 + 1\right) + \frac{A_0}{c_1} \quad ...(3.174)$$

The fourth condition (2.62) is verified if

$$t_{15}^* > c_1^6\left(c_1 - 2a_1 + 1\right)^7 + A_6 c_1^5\left(c_1 - 2a_1 + 1\right)^6 + A_5 c_1^4\left(c_1 - 2a_1 + 1\right)^5 + A_4 c_1^3\left(c_1 - 2a_1 + 1\right)^4 + A_3 c_1^2\left(c_1 - 2a_1 + 1\right)^3$$
$$+ A_2 c_1\left(c_1 - 2a_1 + 1\right)^2 + A_1\left(c_1 - 2a_1 + 1\right) + \frac{A_0}{c_1} \quad ...(3.175)$$

The first fourteen integer values of t_1^* to t_{14}^* that satisfy (3.173) and (3.174) and that render $\delta a\left(t_1^*\right)$ to $\delta a\left(t_{14}^*\right)$ (3.172) integers, are related in pairs by (2.64) and can be found from $\delta a\left(t_j^*\right)$, as

$$\delta a\left(t_{15-j}^{*}\right)=c_{1}-2a_{1}+1-\delta a\left(t_{j}^{*}\right)=c_{1}-a_{1}-a_{j+1}+1 \quad \text{...(3.176)}$$

for $1 \leq j \leq 7$. The first group of eight pairs of solutions can now be written as

$$\begin{aligned}
& a_1 < a_2 = a_1 + \delta a\left(t_1^*\right) < a_3 = a_1 + \delta a\left(t_2^*\right) < a_4 = a_1 + \delta a\left(t_3^*\right) \\
& \quad < a_5 = a_1 + \delta a\left(t_4^*\right) < a_6 = a_1 + \delta a\left(t_5^*\right) < a_7 = a_1 + \delta a\left(t_6^*\right) \\
& \quad < a_8 = a_1 + \delta a\left(t_7^*\right) < a_9 = a_1 + \delta a\left(t_8^*\right) = c_1 - a_8 + 1 \\
& \quad < a_{10} = a_1 + \delta a\left(t_9^*\right) = c_1 - a_7 + 1 < a_{11} = a_1 + \delta a\left(t_{10}^*\right) = c_1 - a_6 + 1 \\
& \quad < a_{12} = a_1 + \delta a\left(t_{11}^*\right) = c_1 - a_5 + 1 < a_{13} = a_1 + \delta a\left(t_{12}^*\right) = c_1 - a_4 + 1 \\
& \quad < a_{14} = a_1 + \delta a\left(t_{13}^*\right) = c_1 - a_3 + 1 < a_{15} = a_1 + \delta a\left(t_{14}^*\right) = c_1 - a_2 + 1 \\
& \quad < a_{16} = c_1 - a_1 + 1
\end{aligned} \quad \text{...(3.177)}$$

The other groups of eight pairs of solutions are found by adding $(k-1)c_1$ to the first 16 solutions as

$$a_{16k-15} = a_1 + (k-1)c_1 < \ldots < a_{16(k-1)+j} = a_j + (k-1)c_1 < \ldots < a_{16k} = a_{16} + (k-1)c_1 \quad \text{...(3.178)}$$

for $1 \leq j \leq 16$ and for all positive integers k. The relations for a in (3.177) can be combined into the single general relation (2.78), yielding all values of solutions

$$a_{\left(16k-\frac{15\pm(2j-1)}{2}\right)} = \frac{(1-c_1)\pm\left(2a_{(9-j)}-c_1-1\right)}{2}+kc_1 \quad \text{...(3.179)}$$

with $1 \leq j \leq 8$ and $k > 0$.

3.6.4. Calculation Methods

From the algebraic method, starting with the first non-excluded value of $f_1 = 3$, yielding $c_1 = 103$, one has that the smallest value of f_2 that satisfies (3.152) is $f_{2,1} = 4119183862359$ to which corresponds $K = 8$ as

$$c_1 f_{2,1} + f_1 = 103 \cdot 4119183862359 + 3 = \Delta(2\Delta+1)\,(2\Delta(2\Delta+1)\,((2\Delta)^2(2\Delta+3)\,(2\Delta+7)+2\,(14\Delta+3))+1) = 424275937822980 \quad \text{...(3.180)}$$

where Δ is written here for $\Delta(K=8)$, yielding from (2.8) the first seed solution $a_1 = 9$ for this value of $f_1 = 3$. The first seven values of $\delta a(t^*)$ are $\delta a\left(t_1^*\right) = 3$, $\delta a\left(t_2^*\right) = 5$, $\delta a\left(t_3^*\right) = 10$, $\delta a\left(t_4^*\right) = 17$, $\delta a\left(t_5^*\right) = 20$, $\delta a\left(t_6^*\right) = 24$, and $\delta a\left(t_7^*\right) = 35$, yielding $a_2 = 12$, $a_3 = 14$, $a_4 = 19$, $a_5 = 26$, $a_6 = 29$, $a_7 = 33$, and $a_8 = 44$. From (3.177), the other solutions for $k = 1$, are $a_9 = 60$, $a_{10} = 71$, $a_{11} = 75$, $a_{12} = 78$, $a_{13} = 85$, $a_{14} = 90$, $a_{15} = 92$ and $a_{16} = 95$. Other values are given in Table 10 for $3 \leq f_1 \leq 10$ and $k = 1$ to 3; the corresponding values of f_2 can be found in (Pletser, 2024b). Corresponding composite $G_{Ma,17}$ can be found in Sequence A022533 in (Sloane, 2024).

Note as well that for $f_1 = 3855$, which is the value of $\Phi_{17}(\Delta(K))$ for $K = 1$ (i.e., for $a = 2$), one has $f_2 = 0$ in (3.153), meaning that $a = 2$ does not yield a composite but a generalized Mersenne prime, which is the Mersenne prime $M_{17} = 131071$.

Like for the previous cases, one can also test the ratio

$$R = \frac{\Delta(2\Delta+1)\Big(2\Delta(2\Delta+1)\Big((2\Delta)^2(2\Delta+3)(2\Delta+7)+2(14\Delta+3)\Big)+1\Big)-i}{34i+1} \quad \text{...(3.181)}$$

with here $\Delta = \Delta(a-1)$, to find whether a $GM_{a,17}$ is prime or composite.

Table 10: Values of a_κ Yielding Composite $GM_{a,17}$ for n = 17

k =	$f_1 = 3, c_1 = 103$			$f_1 = 4, c_1 = 137$			$f_1 = 7, c_1 = 239$			$f_1 = 9, c_1 = 307$		
	1	2	3	1	2	3	1	2	3	1	2	3
$a_{\kappa+}$	9	112	215	6	143	280	10	249	488	11	318	625
	12	115	218	15	152	289	33	272	511	35	342	649
	14	117	220	27	164	301	42	281	520	40	347	654
	19	122	225	37	174	311	43	282	521	63	370	677
	26	129	232	55	192	329	49	288	527	77	384	691
	29	132	235	60	197	334	91	330	569	115	422	729
	33	136	239	64	201	338	99	338	577	143	450	757
	44	147	250	65	202	339	105	344	583	145	452	759
$a_{\kappa-}$	60	163	266	73	210	347	135	374	613	163	470	777
	71	174	277	74	211	348	141	380	619	165	472	779
	75	178	281	78	215	352	149	388	627	193	500	807
	78	181	284	83	220	357	191	430	669	231	538	845
	85	188	291	101	238	375	197	436	675	245	552	859
	90	193	296	111	248	385	198	437	676	268	575	882
	92	195	298	123	260	397	207	446	685	273	580	887
	95	198	301	132	269	406	230	469	708	297	604	911

Note: Where $\kappa_+ = (16k - i)$ with $15 \geq i \geq 8$ and $\kappa_- = (16k - j)$ with $7 \geq j \geq 0$, respectively from the first to the eighth row for $a_{\kappa+}$ and $a_{\kappa-}$.

Table 11: f_1, c_1 and First Values of $a_{\kappa+}$ for Prime n, $3 \leq n \leq 17$

n	f_1	c_1	$a_{\kappa+}$	n	f_1	c_1	$a_{\kappa+}$
3	1	7	2	11	1	23	2, 3, 4, 9, 11
	2	13	6		2	45	f_1 excluded value
	3	19	3		3	67	12, 15, 17, 25, 32
	4	25	f_1 excluded value		4	89	2, 7, 24, 31, 38
	5	31	9		5	111	f_1 excluded value
	6	37	4		6	133	f_1 excluded value
	7	43	17		7	155	f_1 excluded value
	8	49	23		8	177	f_1 excluded value
	9	55	f_1 excluded value		9	199	62, 63, 69, 81, 83
	10	61	5		10	221	f_1 excluded value
5	1	11	4, 5	13	1	27	f_1 excluded value
	2	21	f_1 excluded value		2	53	3, 7, 16, 20, 22, 23
	3	31	2, 10		3	79	5, 7, 15, 22, 32, 35
	4	41	9, 12		4	105	f_1 excluded value
	5	51	f_1 excluded value		5	131	4, 19, 21, 31, 59, 63
	6	61	16, 24		6	157	8, 12, 22, 29, 63, 70
	7	71	4, 19		7	183	f_1 excluded value
	8	81	f_1 excluded value		8	209	f_1 excluded value
	9	91	f_1 excluded value		9	235	f_1 excluded value
	10	101	27, 29		10	261	f_1 excluded value

Table 11 (Cont.)

n	f_1	c_1	$a_{\kappa+}$	n	f_1	c_1	$a_{\kappa+}$
7	1	15	f_1 excluded value	17	1	35	f_1 excluded value
	2	29	3, 5, 6		2	69	f_1 excluded value
	3	43	14, 15, 20		3	103	9, 12, 14, 19, 26, 29, 33, 44
	4	57	f_1 excluded value		4	137	6, 15, 27, 37, 55, 60, 64, 65
	5	71	3, 16, 22		5	171	f_1 excluded value
	6	85	f_1 excluded value		6	205	f_1 excluded value
	7	99	f_1 excluded value		7	239	10, 33, 42, 43, 49, 91, 99, 105
	8	113	14, 40, 46		8	273	f_1 excluded value
	9	127	2, 18, 42		9	307	11, 35, 40, 63, 77, 115, 143, 145

Note: Where $\kappa_+ = (n - 1 - i)$ with $(n - 2) \geq i \geq \frac{n-1}{2}$.

4. Conclusion

It was shown that with the proposed generalization of Mersenne numbers, the distribution of composite Generalized Mersenne numbers follow simple laws demonstrated in three theorems, as composite $GM_{a,n}$ appear periodically in an infinite number of groups of pairs of solutions in a, embedded into each others. The most remarkable aspect of composite $GM_{a,n}$ is that their distribution is completely characterized once the first values of a yielding composite $GM_{a,n}$ are found, as composite $GM_{a,n}$ are spaced regularly, separated by intervals of values depending on their factors $c_1 = 2nf_1 + 1$. Three methods were presented to calculate composite $GM_{a,n}$ and applied for the first six prime exponents n from 3 to 17. Table 11 summarizes for each exponent n the values of f_1, c_1 and the first $\frac{(n-1)}{2}$ values of a_κ, and those excluded values of f_i and forbidden factors c_i not yielding solutions in a.

For specific non-excluded values of f_1, the first values of a_κ characterize completely the series of values of a yielding composite $GM_{a,n}$ numbers by the relation (2.78) for all positive integers k. For values of f_i producing solutions in a, the corresponding value of c_i is either prime or a composite of the form $(2nj + 1)(2nk + 1)$, with j and k integers, while for f_i to be an excluded values (i.e., to not produce solutions in a), the corresponding forbidden factors c_i are composites of the form $(2nj - 1)(2nk \pm t)$ with t all odd integers from 3 to $(n - 2)$. Prime distributions in Generalized Mersenne numbers are further investigated in a following paper.

Conflict of Interest

The Author declares no conflict of interest.

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