https://doi.org/10.51483/IJPAMR.3.2.2023.33-47 ISSN: 2789-9160

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Research Paper Open Access

A Solution of the Navier-Stokes Problem for an Incompressible Fluid with Cauchy Condition

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Article Info

Volume 3, Issue 2, October 2023
Received : 14 April 2023
Accepted : 29 August 2023
Published : 05 October 2023
doi: 10.51483/IJPAMR.3.2.2023.33-47

Abstract

The main object of this work is to establish the existence and uniqueness of a solution to the 3D Navier-Stokes (NS) system for an incompressible fluid with viscosity. The nonlinearity of the NS system, as well as the need to estimate velocity and pressure for every value of the viscosity parameter make them challenging to solve. In this regard, in the present work, we study the Navier-Stokes system, which describe the flow of a viscous incompressible fluid and the solution was obtained for velocity and pressure in an analytical form. In addition, the found pressure distribution law, which is described by a Poisson type equation and plays a fundamental role in the theory of Navier-Stokes systems in constructing analytic smooth (conditionally smooth) solutions.

Keywords: *Navier-Stokes Equation (NSE), Differential Equation (DE), Pressure, incompressible fluid, Poisson equation, Solution uniqueness*

1. Introduction

In this work, we are not trying to consider the extensive references on the Navier-Stokes system, since there are fundamental works in this area (Landau and Lifshith , 1987; Prantdl, 1961; Schlichting, 1974) and others. Some special of the above problems were also investigated in the works (Scheffer, 1976; Beale *et al.*, 1984; Fernández-Dalgo and Lemarié-Rieusset , 2021; Marcati and Schwab, 2020) etc.

It is known that the methods of integral transformations in the theory of partial differential equations made it possible to find solutions to many problems (Sobolev, 1966; Friedman , 1958) and clarify the physical meaning of some basic laws and phenomena in fluid mechanics. Therefore, this paper presents one of the developed transformations of the said species.

In this regard, in the present work, we study the Navier-Stokes system, which describe the flow of a viscous incompressible fluid filling all of R^3, i.e.:

$$\frac{\partial v}{\partial t}+(v\nabla)v=f-\frac{1}{\rho}\nabla P+\mu\Delta v, \quad ...(1.1)$$

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$$\operatorname{div} \nu = 0, \qquad ...(1.2)$$

with initial conditions

$$\nu\big|_{t=0} = \psi(x),\ \forall x \in R^3, \qquad ...(1.3)$$

where $R^3 \ni \psi(x)$ is the known initial velocity vector, $R^3 \ni f(x,t)$ is external applied force (e.g. gravity), $0 < \mu$ is kinematic viscosity, ρ is density, Δ is Laplace operator, ∇ is Hamilton operator. These equations are to be solved for an unknown velocity vector $\nu \in R^3$ and pressure $P(x, t)$, and Equation (2) just says that the fluid is incompressible.

1.1. Aim of Research

The main object of this work is to establish the existence and uniqueness of a solution to the Navier-Stokes system for an incompressible fluid and at the same time, it is proved that:

a) The solutions of the transformed equations are regular with respect to the viscosity coefficient μ, and they simplify the analysis of the original problem,

b) The found pressure distribution law, which is described by a Poisson type equation and plays a fundamental role in the theory of Navier-Stokes systems in constructing analytic conditionally smooth (smooth) solutions.

c) At the same time, the obtained results meet the requirements of the "Navier-Stokes Millennium problem-NSMP" (Fefferman, 2000).

In the introduced space $G^1_{3,h}(D_0)$, the norm is defined as:

$$\begin{cases} \|\nu\|_{G^1_{3,h}(D_0)} = \sum\limits_{i=1}^{3}\|\nu_i\|_{G^1_h(D_0)} = \sum\limits_{i=1}^{3}\{\sum\limits_{0\le|k|\le2}\|D^k\nu_i\|_{C(D\ )} + \|\nu_{it}\|_{1h)}\}, \\ \|\nu_t\|_{1h} = \sup\limits_{R^3}\int\limits_0^\infty h(s)|\nu_t(x,s)|ds;\ h\in L^1(0,\infty), 0\le h\le h_1 = const < \infty, \\ \int\limits_0^\infty h(s)ds \le h_2 = const < \infty, (h_0 = \max(h_1,h_2); D = R^3\times R_+; D_0 = R^3\times(0,\infty), \end{cases}$$

where $k = (k_1, k_2, k_3)$ is the multi-index,

$$\begin{cases} \nu = (\nu_1,\nu_2,\nu_3),\ k = 0 : D^0\nu_i \equiv \nu_i; k \ne 0 : D^k\nu_i = \dfrac{\partial^{|k|}\nu_i}{\partial x_1^{k_1}\partial x_2^{k_2}\partial x_3^{k_3}},\ (i=\overline{1,3}), \\ |k| = \sum\limits_{j=1}^{2} k_j,\ (k_j = 0,1,2; j = 1,2). \end{cases} \qquad ...(1.4)$$

A: In this regard, we note that in our early works, for example, in Omurov (2019), we proposed method for constructing smooth (conditionally smooth) solutions of 3D Navier-Stokes equations in $G^1_3(D_1 = R^3\times(0,T_0))$:

$$\begin{cases} \|\nu\|_{G^1_3(D_1)} = \sum\limits_{i=1}^{3}\|\nu_i\|_{G^1(D_1)} = \sum\limits_{i=1}^{3}\{\sum\limits_{0\le|k|\le2}\|D^k\nu_i\|_{C(\overline{D_1})} + \|\nu_{it}\|_{1)}\}, \\ \|\nu_t\|_1 = \sup\limits_{R^3}\int\limits_0^{T_0} |\nu_t(x,s)|ds \end{cases}$$

with the condition:

$$\mathrm{v}\big|_{t=0} = \varphi(x)\lambda, \forall x \in R^3, \qquad ...(1.5)$$

where $\varphi(x)$ is known scalar function, $R^3 \ni \lambda$ is given vector with positive constant components: $0 < \lambda_i, (i = \overline{1,3})$. Since it takes place

$$\begin{cases} f \in R^3,\ \varphi \in R,\ \lambda \in R^3 : \\ \mathrm{div} f = 0;\ \mathrm{div}(\lambda\varphi) = 0, \\ \left|D^k \varphi\right| \le \beta_0 = const,\ \forall x \in R^3, \\ \left|D^k f_i\right| \le \beta_1 = const,\ \forall (x,t) \in \bar{D}_1,\ (i = \overline{1,3}), \end{cases} \qquad ....(1.6)$$

then, we seek the solution of the Navier-Stokes problem in the form:

$$\mathrm{v} = \theta\lambda + \mu J(x,t), \qquad ...(1.7)$$

here $R^3 \in J$ is known vector-valued function of the form:

$$\begin{cases} J = \dfrac{1}{2^3\sqrt{\pi^3}} \displaystyle\int_0^t \int_{R^3} \dfrac{1}{\sqrt{(\mu(t-s))^3}} f(\tau,s) \exp(-\dfrac{r^2}{4\mu(t-s)}) d\tau ds, (x,\tau \in R^3), \\ 0 < \mu < 1;\ r = |x-\tau| = \sqrt{\displaystyle\sum_{i=1}^{3} (x_i - \tau_i)^2};\ J\big|_{t=0} = 0, \forall x \in R^3, \\ G(x,\tau,t-s) \equiv \begin{cases} \dfrac{1}{2^3(\sqrt{\mu\pi(t-s)})^3} \exp(-\dfrac{|x-\tau|^2}{4\mu(t-s)}), (t > s), \\ 0, (t \le s), \end{cases} \\ L[G] \equiv \dfrac{\partial G}{\partial t} - \mu\Delta G = 0, \end{cases} \qquad ...(1.8)$$

and $\theta(x,t)$ is a new unknown scalar function with the condition:

$$\theta\big|_{t=0} = \varphi(x),\ \forall x \in R^3. \qquad ...(1.9)$$

Lemma 1: In case of (1.7), when conditions (1.2), (1.6) are satisfied, the inertial terms of Equation (1.1), taking into account (1.7), are linearized with respect to the introduced function $\theta(x,t)$ and its derivatives.

Proof: In fact, under conditions (1.2) and (1.6), it follows from (1.7):

$$\begin{cases} \mathrm{div} f = 0;\ \mathrm{div} J = \dfrac{1}{\sqrt{\pi^3}} \displaystyle\int_0^t \int_{R^3} \mathrm{div} f(x + 2\xi\sqrt{\mu(t-s)}, s) \exp(-|\xi|^2) d\xi ds = 0, \\ \tau = x + 2\xi\sqrt{\mu(t-s)} \in R^3;\ 0 = \mathrm{div}\ \mathrm{v} = \mathrm{div}\theta\lambda + \mu \mathrm{div} J = \displaystyle\sum_{i=1}^{3} \lambda_i \theta_{x_i}. \end{cases} \qquad ...(1.10)$$

And since

$$(\theta\lambda\nabla)\theta\lambda = \lambda_i\theta(\sum_{j=1}^{3} \lambda_j\theta_{x_j}) = 0, (i = \overline{1,3}), \qquad ...(1.11)$$

then, on the basis of (1.7), (1.10) and (1.11), the inertial terms of Equation (1.1) are equivalently converted to the form:

$$(\nu\nabla)\nu = (\theta\lambda\nabla)\theta\lambda + \mu[(\theta\lambda\nabla)J + (J\nabla)\theta\lambda] + \mu^2(J\nabla)J = \mu[(\theta\lambda\nabla)J + (J\nabla)\theta\lambda] + \mu^2(J\nabla)J. \quad ...(1.12)$$

So this means that under condition (1.2), the inertial terms of Equation (1.1), taking into account (1.7), are linearized with respect to the newly introduced function $\theta(x,t)$ and its derivatives with respect to $x \in R^3$, and the nonlinearity goes over to the known vector of the function $J(x,t)$ and partial derivatives with respect to $x \in R^3$. Which was required to show.

Further, substituting transformation (1.7) into NSE (1.1), we obtain a linear inhomogeneous differential equation of the type of heat conduction with variable coefficients:

$$\frac{\partial\theta}{\partial t}\lambda + \mu[(\theta\lambda\nabla)J + (J\nabla)\theta\lambda] + \mu^2(J\nabla)J = (1-\mu)f - \frac{1}{\rho}\nabla P + (\mu\Delta\theta)\lambda, \quad ...(1.13)$$

in this case,

$$\begin{cases} \lambda_1^{-1}\sum_{j=1}^{3}\lambda_j J_{1x_j} \equiv \lambda_2^{-1}\sum_{j=1}^{3}\lambda_j J_{2x_j} \equiv \lambda_3^{-1}\sum_{j=1}^{3}\lambda_j J_{3x_j}, \\ \lambda_1^{-1}\{\frac{1}{\rho}P_{x_1} - f_1(1-\mu) + \mu^2\sum_{j=1}^{3}J_j J_{1x_j}\} \equiv \lambda_2^{-1}\{\frac{1}{\rho}P_{x_2} - f_2(1-\mu) + \mu^2\sum_{j=1}^{3}J_j J_{2x_j}\} \equiv \\ \equiv \lambda_3^{-1}\{\frac{1}{\rho}P_{x_3} - f_3(1-\mu) + \mu^2\sum_{j=1}^{3}J_j J_{3x_j}\}, \end{cases} \quad ...(1.14)$$

(1.14) is the condition of unequivocal compatibility for case (1.13), since θ is a scalar function.

Remark 1: The remark consists of two parts related to formula (1.7) and (1.14).

a: In transformation (1.7) it is assumes that: $divf = 0$. But this transformation can also be introduced in the case when: $divf \neq 0$. For this purpose, let us introduce the vector function $f^0(x,t),(x \in R^3)$:

$$\begin{cases} f^0 = (f_1(0,x_2,x_3,t)), f_2(x_1,0,x_3,t), f_3(x_1,x_2,0,t)), \\ divf^0 = \frac{\partial}{\partial x_1}f_1(0,x_2,x_3,t) + \frac{\partial}{\partial x_2}f_2(x_1,0,x_3,t) + \frac{\partial}{\partial x_3}f_3(x_1,x_2,0,t) = 0. \end{cases} \quad ...(0.1)$$

Then in this case, the vector function $J(x,t)$ is represented as:

$$\begin{cases} J^0 = (J_1^0(0,x_2,x_3,t)), J_2^0(x_1,0,x_3,t), J_3^0(x_1,x_2,0,t)), \\ J_i^0\big|_{x_i=0} = \frac{1}{2^3\sqrt{\pi^3}}\int_0^t\int_{R^3}(\exp(-\frac{r^2}{4\mu(t-s)}))f_i(\tau,s)\big|_{\tau_i=0}\frac{d\tau ds}{\sqrt{(\mu(t-s))^3}}, (i=\overline{1,3};\ \tau \in R^3), \\ J^0\big|_{t=0} = 0, \forall x \in R^3, \\ divJ^0 = \frac{\partial}{\partial x_1}J_1^0 + \frac{\partial}{\partial x_2}J_2^0 + \frac{\partial}{\partial x_3}J_3^0 = 0. \end{cases} \quad ...(0.2)$$

Next, we obtain similar results as in the case of lemma 1.

b: To understand (1.14) we give a concrete example from the field of the system of algebraic equations. For this purpose, consider a system with one unknown quantity z, i.e.:

$$\lambda_i z + a_i = \lambda_i b, (i = \overline{1,3}). \qquad ...(0.3)$$

From here we see that if there is

$$\lambda_1^{-1} a_1 = \lambda_2^{-1} a_2 = \lambda_3^{-1} a_3 = a_0 \qquad ...(0.4)$$

then z is uniquely defined in the form

$$z = b - a_0. \qquad ...(0.5)$$

But z, can be defined differently, i.e.:

$$z = b - (\lambda_1 + \lambda_2 + \lambda_3)^{-1} (a_1 + a_2 + a_3) \qquad ...(0.6)$$

or with respect to Equation (0.6), performing some mathematical transformation we obtain

$$z = b - (\lambda_1 + \lambda_2 + \lambda_3)^{-1} (\lambda_1 \frac{a_1}{\lambda_1} + \lambda_2 \frac{a_2}{\lambda_2} + \lambda_3 \frac{a_3}{\lambda_3}) \underset{(0.5)}{=} b - a_0. \qquad ...(0.7)$$

This means that the first and second paths are equivalent. So, under condition (0.4), z is indeed uniquely determined from (0.3). Which was required to show.

Note that condition (1.14) is an analogue of a condition like (0.4) for the system with scalar unknown. Therefore, when we solve the system Equation (1.13) we choose the second path as shown in the case (0.6).

Next, the equation for the pressure is derived:

$$\begin{cases} \frac{1}{\rho} \Delta P = -\sum_{i=1}^{3} \sum_{j=1}^{3} \mathrm{v}_{ix_j} \mathrm{v}_{jx_i} = -\{F_0 + \mu[\sum_{i=1}^{3} (\sum_{j=1}^{3} \lambda_j J_{ix_j}) \theta_{x_i} + \sum_{i=1}^{3} (\sum_{j=1}^{3} J_{jx_i} \theta_{x_j}) \lambda_i]\}, \\ F_0 \equiv \mu^2 \sum_{i=1}^{3} \sum_{j=1}^{3} J_{ix_j} J_{jx_i}. \end{cases} \qquad ...(1.15)$$

We are taking into account the operation div with respect to Equation (1.13), (that's tantamount to applying the operation div with respect to NSE (1.1), since (1.1) is equivalently converted to the form Equation (1.13) based on transformation (1.7)), since takes places:

$$\begin{cases} \mathrm{div} f = 0, \ \mathrm{div}(\theta_t \lambda) = 0; \ \mathrm{div}(\mu \Delta \theta) \lambda = 0, \ \mathrm{div}(\Delta J) = 0, \ (\mathrm{div} J = 0), \\ \mathrm{div}\{\mu[(\theta \lambda \nabla) J + (J \nabla) \theta \lambda] + \mu^2 (J \nabla) J\} = F_0 + \mu(\sum_{i=1}^{3} (\sum_{j=1}^{3} \lambda_j J_{ix_j}) \theta_{x_i} + \sum_{i=1}^{3} (\sum_{j=1}^{3} J_{jx_i} \theta_{x_j}) \lambda_i). \end{cases}$$

Here, formula (1.15) modifies the Landau – Lifshitz formula (Landau and Lifshith, 1987) and is an of Poisson type. Then it follows from Equation (1.15):

$$\begin{cases} P(x,t) = \int_{R^3} \frac{1}{r} \rho \Omega(\tau,t) d\tau, (x, \tau \in R^3), \\ r = |x - \tau|; \ \Omega(x,t) \equiv \frac{1}{4\pi} \{F_0 + \mu[\sum_{i=1}^{3} (\sum_{j=1}^{3} \lambda_j J_{ix_j}) \theta_{x_i} + \sum_{i=1}^{3} (\sum_{j=1}^{3} J_{jx_i} \theta_{x_j}) \lambda_i]\}, \end{cases} \qquad ...(1.16)$$

at that

$$\frac{\partial}{\partial x} P = \int_{R^3} \rho \Omega(\tau,t) \frac{\partial \frac{1}{r}}{\partial x} d\tau = \int_{R^3} \rho \Omega(\tau,t) \frac{\tau - x}{r^3} d\tau, (\tau - x \in R^3), \qquad ...(1.17)$$

where Equation (1.16) is called the Newtonian potential (Sobolev, 1966). On the other hand, a solution to the Poisson Equation (1.15) tending to zero at infinity will be unique if the function $\theta_{x_i},(i=1,2,3)$ is unique, since the function $\Omega(x,t)$ contains these functions.

To prove the above, we note that the obtained pressure distribution law allows us to express the velocity in integral form when $v \in R^3$. In fact, substituting transformation (1.17) into Equation (1.13) with allowance for condition (1.14), we obtain an inhomogeneous linear integro-differential heat conduction equation with the Cauchy condition:

$$\begin{cases} \theta_t = \Phi + \mu B[\theta,\theta_{x_1},\theta_{x_2},\theta_{x_3}] + \mu\Delta\theta, \\ \theta\big|_{t=0} = \varphi(x), \forall x \in R^3, \end{cases} \qquad ...(1.18)$$

here the known functions contained in DE (1.18) are introduced on the basis of the notation:

$$\begin{cases} \Phi \equiv \Phi_1 + \Phi_2;\ \Phi_1 \equiv d_0^{-1}\sum_{i=1}^{3}(1-\mu)f_i(x,t); d_0 = \sum_{i=1}^{3}\lambda_i > 0, \\ \Phi_2(x,t) \equiv d_0^{-1}[-\mu^2\sum_{i=1}^{3}(\sum_{j=1}^{3}J_j J_{ix_j}) - \frac{1}{4\pi}\int_{R^3}\sum_{i=1}^{3}\frac{\tilde{\xi}_i}{r_1^3}F_0(x+\tilde{\xi};t)d\tilde{\xi}], \\ B[\theta,\theta_{x_1},\theta_{x_2},\theta_{x_3}] \equiv -\{d_0^{-1}\theta(.)\sum_{i=1}^{3}(\sum_{j=1}^{3}\lambda_j I_{ix_j}) + \sum_{j=1}^{3}\theta_{x_j}(.)I_j(.) + \\ +d_0^{-1}(\frac{1}{4\pi}\int_{R^3}\sum_{k=1}^{3}\frac{\tilde{\xi}_k}{r_1^3}[\sum_{i=1}^{3}(\sum_{j=1}^{3}\lambda_j I_{ih_j}(x+\tilde{\xi},t))\theta_{h_i}(x+\tilde{\xi},t) + \sum_{i=1}^{3}(\sum_{j=1}^{3}I_{jh_i}(x+ \\ +\tilde{\xi},t)\theta_{h_j}(x+\tilde{\xi},t))\lambda_i]d\tilde{\xi})\},(r_1 = \sqrt{(\tilde{\xi}_1^2+\tilde{\xi}_2^2+\tilde{\xi}_3^2)^3};\ \ h = x+\tilde{\xi} \in R^3). \end{cases} \qquad ...(1.19)$$

As a result, problem (1.18) is transformed to a system of Volterra and Volterra-Abel equations of second kind, where the solvability of this problem in $G^1(D_1)$ follows from the solvability of this system. Therefore, we obtain similar conclusions for problem (1.1)-(1.3) in $G_3^1(D_1)$.

B: Similar issues were investigated in Omurov (2021), i.e., NSE (1.1), (1.2) with the condition:

$$v\big|_{t=0} = 0,\ \forall x \in R^3, t \in R_+ = [0,\infty), \qquad ...(1.20)$$

at that $f_i(x,t)$ is the component of a given external force f admits the conditions:

$$\begin{cases} f = (f_1,f_2,f_3),\ \operatorname{div} f = 0, \\ \left|D^k f_i\right| \le \beta_1(1+t)^{-q} \le \beta_1 = const,\ ((x,t) \in D = R^3 \times R_+); q = const > 1), \\ \int_0^\infty\int_{R^3} f(x,t)dxdt = \lambda,\ (\lambda \in R^3:\ 0 < \lambda_i = const,\ i = \overline{1,3}), \end{cases} \qquad ...(1.21)$$

and this means that $R^3 \ni \lambda$ is a vector with constant components: $0 < \lambda_i,(i=1,2,3)$, therefore, it becomes possible to use modification of the method (1.7) of the previous sector, i.e.:

$$v = \theta\lambda + \mu(1+t)^{-q}J(x,t), \qquad ...(1.22)$$

where conditions of the form (1.8) are taken into account. At that $\theta(x,t)$ is a new unknown scalar function with the condition:

$$\theta\big|_{t=0} = 0,\ \forall x \in R^3. \qquad ...(1.23)$$

Here transformation (1.22) differs from transformation (1.7), since $t \in R_+$, therefore as a multiplier function, we introduced:

$$\Omega_0(t) \equiv (1+t)^{-q}.$$

Hence, follow conditions (1.10), (1.11) we have

$$\begin{cases} \mathrm{div} f = 0: \\ \mathrm{div} J = \dfrac{1}{\sqrt{\pi^3}} \displaystyle\int_0^t \int_{R^3} \mathrm{div} f(x + 2\xi\sqrt{\mu(t-s)}, s)\exp(-|\xi|^2)d\xi ds = 0, \\ \tau = x + 2\xi\sqrt{\mu(t-s)} \in R^3;\ \mathrm{div}\, \mathrm{v} = 0: \\ 0 = \mathrm{div}\, \mathrm{v} = \mathrm{div}\theta\lambda + \mu(1+t)^{-q}\mathrm{div} J = \displaystyle\sum_{i=1}^{3} \lambda_i \theta_{x_i} \end{cases} \qquad ...(1.24)$$

and

$$(\theta\lambda\nabla)\theta\lambda = \lambda_i\theta(\sum_{j=1}^{3}\lambda_j\theta_{x_j}) = 0, (i = \overline{1,3}). \qquad ...(1.25)$$

Then, taking into account conditions (1.20), (1.24) and (1.25), the inertial terms of Equation (1.1) are equivalently converted to the form:

$$\begin{aligned} (\mathrm{v}\nabla)\mathrm{v} &= (\theta\lambda\nabla)\theta\lambda + \mu(1+t)^{-q}[(\theta\lambda\nabla)J + (J\nabla)\theta\lambda] + \mu^2(1+t)^{-2q}(J\nabla)J = \\ &= \mu(1+t)^{-q}[(\theta\lambda\nabla)J + (J\nabla)\theta\lambda] + \mu^2(1+t)^{-2q}(J\nabla)J. \end{aligned} \qquad ...(1.26)$$

Constraints on external force of the form condition (1.21) make it possible to simplify the Navier-Stokes problem and transform it into a system of integral equations of the second kind. In fact, on the basis of transformation (1.22) condition (1.26), from NSE (1.1) follows the equation:

$$\begin{aligned} &\frac{\partial\theta}{\partial t}\lambda + \mu(1+t)^{-q}[(\theta\lambda\nabla)J + (J\nabla)\theta\lambda] + \mu^2(1+t)^{-2q}(J\nabla)J = (1-\mu(1+t)^{-q})f + \\ &+\mu q(1+t)^{-(q+1)}J - \frac{1}{\rho}\nabla P + (\mu\Delta\theta)\lambda, \end{aligned} \qquad ...(1.27)$$

since θ is a scalar function, then the condition:

$$\begin{cases} \lambda_1^{-1}\displaystyle\sum_{j=1}^{3}\lambda_j J_{1x_j} \equiv \lambda_2^{-1}\sum_{j=1}^{3}\lambda_j J_{2x_j} \equiv \lambda_3^{-1}\sum_{j=1}^{3}\lambda_j J_{3x_j}, \\ \lambda_1^{-1}\{\dfrac{1}{\rho}P_{x_1} - f_1(1-\mu(1+t)^{-q}) - \mu q(1+t)^{-(q+1)}J_1 + \mu^2(1+t)^{-2q}\displaystyle\sum_{j=1}^{3}J_j J_{1x_j}\} \equiv \\ \equiv \lambda_2^{-1}\{\dfrac{1}{\rho}P_{x_2} - f_2(1-\mu(1+t)^{-q}) - \mu q(1+t)^{-(q+1)}J_2 + \mu^2(1+t)^{-2q}\displaystyle\sum_{j=1}^{3}J_j J_{2x_j}\} \equiv \\ \equiv \lambda_3^{-1}\{\dfrac{1}{\rho}P_{x_3} - f_3(1-\mu(1+t)^{-q}) - \mu q(1+t)^{-(q+1)}J_3 + \mu^2(1+t)^{-2q}\displaystyle\sum_{j=1}^{3}J_j J_{3x_j}\}, \end{cases} \qquad ...(1.28)$$

is the condition of unequivocal compatibility for case Equation (1.27). From where the equation for pressure is derived:

$$\begin{cases} \frac{1}{\rho}\Delta P = -\sum_{i=1}^{3}\sum_{j=1}^{3} \nu_{ix_j}\nu_{jx_i} = -\{F_0 + \mu(1+t)^{-q}[\sum_{i=1}^{3}(\sum_{j=1}^{3}\lambda_j J_{ix_j})\theta_{x_i} + \sum_{i=1}^{3}(\sum_{j=1}^{3} J_{jx_i}\theta_{x_j})\lambda_i]\}, \\ F_0 \equiv \mu^2(1+t)^{-2q}\sum_{i=1}^{3}\sum_{j=1}^{3} J_{ix_j}J_{jx_i}, \end{cases} \quad ...(1.29)$$

where

$$\begin{cases} \mathrm{div} f = 0;\ \mathrm{div}(\theta_t\lambda) = 0;\ \mathrm{div}(\mu\Delta\theta)\lambda = 0; \mathrm{div} J = 0, \\ \mathrm{div}\{\mu(1+t)^{-q}[(\theta\lambda\nabla)J + (J\nabla)\theta\lambda] + \mu^2(1+t)^{-2q}(J\nabla)J\} = \mu(1+t)^{-q}(\sum_{i=1}^{3}(\sum_{j=1}^{3}\lambda_j J_{ix_j})\theta_{x_i} + \\ +\sum_{i=1}^{3}(\sum_{j=1}^{3} J_{jx_i}\theta_{x_j})\lambda_i) + F_0. \end{cases}$$

On the other hand, we note that it follows from Equation (1.29):

$$\begin{cases} \Omega(x,t) \equiv \frac{1}{4\pi}\{F_0 + \mu(1+t)^{-q}[\sum_{i=1}^{3}(\sum_{j=1}^{3}\lambda_j J_{ix_j})\theta_{x_i} + \sum_{i=1}^{3}(\sum_{j=1}^{3} J_{jx_i}\theta_{x_j})\lambda_i]\}, \\ P(x,t) = \int_{R^3}\frac{1}{r}\rho\Omega(\tau,t)d\tau, (x,\tau \in R^3,\ r = |x-\tau|), \end{cases} \quad ...(1.30)$$

here Equation (1.30) tends to zero at infinity, and there are second-order partial continuous derivatives, and for the first-order partial derivatives it takes place:

$$\frac{\partial}{\partial x}P = \int_{R^3}\rho\Omega(\tau,t)\frac{\partial\frac{1}{r}}{\partial x}d\tau = \int_{R^3}\rho\Omega(\tau,t)\frac{\tau - x}{r^3}d\tau, (\tau - x \in R^3). \quad ...(1.31)$$

Therefore, excluding pressure from Equation (1.27), we obtain a linear differential equation with variable coefficients and with the Cauchy condition of the form:

$$\begin{cases} \theta_t = \Phi + \mu(1+t)^{-q}\zeta(x,t) + \mu\Delta\theta, \\ \theta|_{t=0} = 0,\ \forall x \in R^3, \\ \zeta(x,t) = -\{d_0^{-1}\theta(.)\sum_{i=1}^{3}(\sum_{j=1}^{3}\lambda_j I_{is_j}(.)) + \sum_{j=1}^{3}\theta_{x_j}(.)I_j(.) + d_0^{-1}(\frac{1}{4\pi}\int_{R^3}\sum_{k=1}^{3}\frac{\tilde{\xi}_k}{r_1^3}[\sum_{i=1}^{3}(\sum_{j=1}^{3}\lambda_j I_{ih_j}(x+ \\ +\tilde{\xi},t))\theta_{h_i}(x+\tilde{\xi},t) + \sum_{i=1}^{3}(\sum_{j=1}^{3} I_{jh_i}(x+\tilde{\xi},t)\theta_{h_j}(x+\tilde{\xi},t))\lambda_i]d\tilde{\xi})\},\ (h = x+\tilde{\xi} \in R^3), \end{cases} \quad ...(1.32)$$

where

$$\begin{cases} \Phi \equiv \Phi_1 + \Phi_2;\ \Phi_1 \equiv d_0^{-1}\sum_{i=1}^{3}[(1-\mu(1+t)^{-q})f_i(x,t) + \mu q(1+t)^{-(q+1)}J_i], \\ \Phi_2(x,t) \equiv d_0^{-1}[-\mu^2(1+t)^{-2q}\sum_{i=1}^{3}(\sum_{j=1}^{3} J_j J_{ix_j}) - \frac{1}{4\pi}\int_{R^3}\sum_{i=1}^{3}\frac{\tilde{\xi}_i}{r_1^3}F_0(x+\tilde{\xi};t)d\tilde{\xi}] = \\ = \mu^2 d_0^{-1}[-(1+t)^{-2q}\sum_{i=1}^{3}(\sum_{j=1}^{3} J_j J_{ix_j}) - \frac{1}{4\pi}\int_{R^3}\sum_{i=1}^{3}\frac{\tilde{\xi}_i}{r_1^3}(1+t)^{-2q} \times \\ \times\sum_{m=1}^{3}\sum_{k=1}^{3}(J_{mx_k}(x+\tilde{\xi};t)J_{kx_m}(x+\tilde{\xi};t))],\ (r_1 = \sqrt{(\tilde{\xi}_1^2 + \tilde{\xi}_2^2 + \tilde{\xi}_3^2)^3};\ d_0 = \sum_{i=1}^{3}\lambda_i > 0). \end{cases} \quad (1.33)$$

In such an approach the solution of problem is reduced to finding two functions, $\theta(x,t)$ and $\zeta(x,t)$. The latter can be usually determined without difficulties and we will solve the equation with respect to $\zeta(x,t)$ by the Picard's method (Sobolev, 1966). Besides, since the function θ has continuous partial derivatives up to the second order inclusive with respect to spatial coordinates, and a first order time derivative, then problem (1.32) with sufficiently smooth data is solvable in $W^0(D)$. Therefore, based on transformation (1.22) we have similar conclusions for problem (1.1) to (1.3) we obtain in $W_3^0(D)$:

$$\begin{cases} \nu \in R^3, \\ \|\nu\|_{W_3^0(D)} = \sum_{i=1}^{3}\|\nu_i\|_{W^0(D)} = \sum_{i=1}^{3}\{\sum_{0\le|k|\le 2}\|D^k \nu_i\|_{C(D)} + \|\nu_{it}\|_{C(D)}\}. \end{cases}$$

2. Fluid with the Cauchy Condition (1.3)

Methods of analysis of physical phenomena are based on statements of corresponding mathematical problems formulated by means of various kinds of functional equations and certain additional conditions. The solutions of these problems may be considered the main aim of the theoretical investigation.

Let is the velocity vector satisfies conditions (1.2), (1.3) and takes place

$$\int_{R^3}\psi(\tau)d\tau + \int_0^\infty\int_{R^3} f(\tau,s)d\tau ds = \lambda,\ (f \in R^3, \tau \in R^3, D_0 = R^3\times(0,\infty), D = R^3 \times R_+), \quad ...(2.1)$$

where $R^3 \ni \lambda$ is a known vector with positive constant components: $0<\lambda_i, (i=\overline{1,3})$, and then applying the transformation:

$$\nu = \theta\lambda + (\exp(-\frac{t}{\mu\delta_0}))J(x,t), \quad ...(2.2)$$

where $\theta(x,t)$ is the new unknown scalar function, and $0<\delta_0$ is the introduced constant, which ensures the application of the Banach principle and the Picard's method for the system of integral equations of Volterra-Abel type of the second kind, into which the original problem is transformed, $R^3 \ni J(x,t)$ is the given vector:

$$\begin{cases} J = \frac{1}{2^3(\sqrt{\mu\pi t})^3}\int_{R^3}\psi(\tau)\exp(-\frac{|x-\tau|^2}{4\mu t})d\tau, (|x-\tau| = \sqrt{\sum_{i=1}^{3}(x_i-\tau_i)^2};\ x,\tau\in R^3), \\ J|_{t=0} = \psi(x),\ \forall x \in R^3, (\ 0<\mu<1;\ \ 0<\delta_0 = const < 1), \\ G(x,\tau,t) \equiv \frac{1}{2^3(\sqrt{\mu\pi t})^3}\exp(-\frac{|x-\tau|^2}{4\mu t}), (t>0), \\ L[G] \equiv \frac{\partial G}{\partial t} - \mu\Delta G = 0, \end{cases} \quad ...(2.3)$$

where $\theta(x,t)$ is a new unknown scalar function with the condition:

$$\theta(x,0) = 0, \forall x \in R^3, \quad ...(2.4)$$

at that

$$\begin{cases} \text{div}\ \nu = 0,\ (\text{div}\,\psi = 0):\ \ \text{div}J = \frac{1}{\sqrt{\pi^3}}\int_{R^3}\exp(-|\xi|^2)\text{div}\,\psi(x+2\xi\sqrt{\mu t})d\xi = 0, \\ \tau = x + 2\xi\sqrt{\mu t} \in R^3;\ \ 0 = \text{div}\ \nu = \text{div}(\theta\lambda) + (\exp(-\frac{t}{\mu\delta_0}))\text{div}J = \text{div}(\theta\lambda) = \sum_{i=1}^{3}\theta_{x_i}\lambda_i. \end{cases} \quad ...(2.5)$$

Then, taking into account conditions (1.11) and (2.5), the inertial terms of Equation (1.1) are equivalently converted to the form:

$$(\mathrm{v}\nabla)\mathrm{v} = (\theta\lambda\nabla)\theta\lambda + (\exp(-\frac{t}{\mu\delta_0}))[(\theta\lambda\nabla)J + (J\nabla)\theta\lambda] + (\exp(-\frac{2t}{\mu\delta_0}))(J\nabla)J =$$
$$= (\exp(-\frac{t}{\mu\delta_0}))[(\theta\lambda\nabla)J + (J\nabla)\theta\lambda] + (\exp(-\frac{2t}{\mu\delta_0}))(J\nabla)J. \quad ...(2.6)$$

The conditions of the form (2.6) make it possible to simplify the Navier-Stokes problem and transform it into a system of Volterian type integral equations of the second kind. In fact, on the basis of transformation (2.2) and (2.6), from NSE (1.1) follows the equation:

$$\frac{\partial\theta}{\partial t}\lambda + (\exp(-\frac{t}{\mu\delta_0}))[(\theta\lambda\nabla)J + (J\nabla)\theta\lambda] = \frac{1}{\mu\delta_0}(\exp(-\frac{t}{\mu\delta_0}))J - (\exp(-\frac{2t}{\mu\delta_0}))\times$$
$$\times(J\nabla)J + f - \rho^{-1}\nabla P + (\mu\Delta\theta)\lambda, \quad ...(2.7)$$

since θ is a scalar function, then the condition:

$$\begin{cases} \lambda_1^{-1}J_1 \equiv \lambda_2^{-1}J_2 \equiv \lambda_3^{-1}J_3, \\ \lambda_1^{-1}\{\frac{1}{\rho}P_{x_1} - f_1 + (exp(-\frac{t}{\mu\delta_0}))(\sum_{j=1}^{3}\lambda_j J_{1x_j} + (exp(-\frac{t}{\mu\delta_0}))\sum_{j=1}^{3}J_j J_{1x_j})\} \equiv \\ \equiv \lambda_2^{-1}\{\frac{1}{\rho}P_{x_2} - f_2 + (exp(-\frac{t}{\mu\delta_0}))(\sum_{j=1}^{3}\lambda_j J_{2x_j} + (exp(-\frac{t}{\mu\delta_0}))\sum_{j=1}^{3}J_j J_{2x_j})\} \equiv \\ \equiv 3\lambda_3^{-1}\{\frac{1}{\rho}P_{x_3} - f_3 + (exp(-\frac{t}{\mu\delta_0}))(\sum_{j=1}^{3}\lambda_j J_{3x_j} + (exp(-\frac{t}{\mu\delta_0}))\sum_{j=1}^{3}J_j J_{3x_j})\}, \end{cases} \quad ...(2.8)$$

is a univocal compatibility condition for Equation (2.7). In addition, the Poisson equation for pressure is derived in the form:

$$\frac{1}{\rho}\Delta P = -\sum_{i=1}^{3}\sum_{k=1}^{3}\mathrm{v}_{ix_k}\mathrm{v}_{kx_i} = -\{F_0 + (\exp(-\frac{t}{\mu\delta_0}))[\sum_{i=1}^{3}(\sum_{j=1}^{3}\lambda_j J_{ix_j})\theta_{x_i} + \sum_{i=1}^{3}(\sum_{j=1}^{3}J_{jx_i}\theta_{x_j})\lambda_i]\}, \quad ...(2.9)$$

and it is obtained on the basis of method (2.2) by applying the operation div to Equation (1.1), (or (2.7)), since

$$\begin{cases} \mathrm{div}f = 0;\ \mathrm{div}J = 0; \mathrm{div}(\theta_t\lambda) = 0;\ \mathrm{div}(\mu\Delta\theta)\lambda = 0; F_0 \equiv exp(-\frac{2t}{\delta_0\mu})\sum_{i=1}^{3}\sum_{j=1}^{3}J_{ix_j}J_{jx_i}, \\ \mathrm{div}\{exp(-\frac{t}{\delta_0\mu})[(\theta\lambda\nabla)J + (J\nabla)\theta\lambda] + exp(-\frac{2t}{\delta_0\mu})(J\nabla)J\} = F_0 + exp(-\frac{t}{\delta_0\mu})\times \\ \times(\sum_{i=1}^{3}(\sum_{j=1}^{3}\lambda_j J_{ix_j})\theta_{x_i} + \sum_{i=1}^{3}(\sum_{j=1}^{3}J_{jx_i}\theta_{x_j})\lambda_i). \end{cases}$$

On the other hand, we note that it follows from Equation (2.9):

$$\begin{cases} \Omega(x,t) \equiv \frac{1}{4\pi}\{F_0 + exp(-\frac{t}{\mu\delta_0})[\sum_{i=1}^{3}(\sum_{j=1}^{3}\lambda_j J_{ix_j})\theta_{x_i} + \sum_{i=1}^{3}(\sum_{j=1}^{3}J_{jx_i}\theta_{x_j})\lambda_i]\}, \\ P(x,t) = \int_{R^3}\frac{1}{r}\rho\Omega(\tau,t)d\tau, (x,\tau \in R^3,\ r = |x-\tau|), \end{cases} \quad ...(2.10)$$

here Equation (2.10) tends to zero at infinity, and there are second-order partial continuous derivatives, and for the first-order partial derivatives it takes place:

$$\frac{\partial}{\partial x}P=\int_{R^3}\rho\Omega(\tau,t)\frac{\partial\frac{1}{r}}{\partial x}d\tau=\int_{R^3}\rho\Omega(\tau,t)\frac{\tau-x}{r^3}d\tau,(\tau-x\in R^3). \qquad ...(2.11)$$

Therefore, excluding pressure from Equation (2.7), we obtain a linear differential equation with variable coefficients and with the Cauchy condition of the form:

$$\begin{cases}\theta_t=\Phi+\zeta(x,t)exp(-\frac{t}{\mu\delta_0})+\mu\Delta\theta,\\ \zeta(x,t)=-\{d_0^{-1}\theta(.)\sum_{i=1}^{3}(\sum_{j=1}^{3}\lambda_j I_{ix_j}(.))+\sum_{j=1}^{3}\theta_{x_j}(.)I_j(.)+d_0^{-1}(\frac{1}{4\pi}\int_{R^3}\sum_{k=1}^{3}\frac{\overline{\xi}_k}{r_1^3}[\sum_{i=1}^{3}(\sum_{j=1}^{3}\lambda_j I_{i\tau_j}(x+\\ +\overline{\xi},t))\theta_{\tau_i}(x+\overline{\xi},t)+\sum_{i=1}^{3}(\sum_{j=1}^{3}I_{j\tau_i}(x+\overline{\xi},t)\theta_{\tau_j}(x+\overline{\xi},t))\lambda_i]d\overline{\xi})\},\ (\tau=x+\overline{\xi}\in R^3),\\ \theta|_{t=0}=0,\forall x\in R^3,\end{cases} \qquad ...(2.12)$$

where

$$\begin{cases}\Phi\equiv\sum_{i=1}^{3}\Phi_i;\ \Phi_1\equiv d_0^{-1}\{\ \sum_{i=1}^{3}f_i(x,t)-(exp(-\frac{2t}{\mu\delta_0}))\sum_{i=1}^{3}(\sum_{j=1}^{3}\ J_jJ_{ix_j})\ \},\\ \Phi_2\equiv d_0^{-1}\left\{\frac{1}{\mu\delta_0}exp(-\frac{t}{\mu\delta_0})\sum_{i=1}^{3}J_i\right\},\\ \Phi_3\equiv d_0^{-1}[-\frac{1}{4\pi}\int_{R^3}\sum_{i=1}^{3}\frac{\overline{\xi}_i}{r_1^3}F_0(x+\overline{\xi};t)d\overline{\xi}]=d_0^{-1}(exp(-\frac{2t}{\mu\delta_0}))[-\frac{1}{4\pi}\int_{R^3}\sum_{i=1}^{3}\frac{\overline{\xi}_i}{r_1^3}\times\\ \times\sum_{m=1}^{3}\sum_{k=1}^{3}(J_{m\tau_k}(x+\overline{\xi};t)J_{k\tau_m}(x+\overline{\xi};t))d\overline{\xi}],(r_1=\sqrt{\sum_{n=1}^{3}\overline{\xi}_n^2};d_0=\sum_{i=1}^{3}\lambda_i>0).\end{cases} \qquad ...(2.13)$$

Further, the solution of the problem under study is reduced to the determination of functions from the equations:

$$\begin{cases}\theta=\Upsilon+\frac{1}{2^3\sqrt{\pi^3}}\int_0^t\int_{R^3}(\exp(-\frac{r^2}{4\mu(t-s)}))(exp(-\frac{s}{\mu\delta_0}))\zeta(\tau,s)\frac{d\tau ds}{(\sqrt{\mu(t-s)})^3}=\Upsilon+\\ +\frac{1}{\sqrt{\pi^3}}\int_0^t\int_{R^3}(\exp(-(|\xi|^2+\frac{s}{\mu\delta_0})))\zeta(x+2\xi\sqrt{\mu(t-s)},s)d\xi ds\equiv(\Gamma\zeta)(x,t),\\ \theta_{x_i}=\Upsilon_{x_i}+\frac{1}{2^3\sqrt{\pi^3}}\int_0^t\int_{R^3}(\exp(-(\frac{r^2}{4\mu(t-s)}+\frac{s}{\mu\delta_0})))\frac{-(x_i-\tau_i)}{2\mu(t-s)}\zeta(\tau,s)\frac{d\tau ds}{(\sqrt{\mu(t-s)})^3}=\\ =\Upsilon_{x_i}+\frac{1}{\sqrt{\pi^3}}\int_0^t\int_{R^3}(\exp(-(|\xi|^2+\frac{s}{\mu\delta_0})))\zeta(x+2\xi\sqrt{\mu(t-s)},s)\frac{\xi_i d\xi ds}{\sqrt{\mu(t-s)}}\equiv\Gamma_i\zeta,(i=\overline{1,3}),\\ \Upsilon_1=\frac{1}{\sqrt{\pi^3}}\int_0^t\int_{R^3}(\exp(-|\xi|^2))(\Phi_1(x+2\xi\sqrt{\mu(t-s)},s)+\Phi_3(x+2\xi\sqrt{\mu(t-s)},s))d\xi ds,\\ \Upsilon_2=\frac{1}{\sqrt{\pi^3}}\int_0^t\int_{R^3}(\exp(-|\xi|^2))\Phi_2(x+2\xi\sqrt{\mu(t-s)},s)d\xi ds,(\zeta(x,t)\in C^{1,0}(D);\Upsilon\equiv\Upsilon_1+\Upsilon_2).\end{cases}$$

...(2.14)

So, taking into account Equations (2.12), (2.14) and $\zeta(x,t)$ we have

$$\begin{cases} \theta=(\Gamma\zeta)(x,t), \\ \zeta=-\{d_0^{-1}(\Gamma\zeta)\sum_{i=1}^{3}(\sum_{j=1}^{3}\lambda_j I_{ix_j}(.))+\sum_{j=1}^{3}(\Gamma_j\zeta)I_j(.)+d_0^{-1}(\frac{1}{4\pi}\int_{R^3}\sum_{k=1}^{3}\frac{\tilde{\xi}_k}{r_1^3}[\sum_{i=1}^{3}(\sum_{j=1}^{3}\lambda_j I_{i\tau_j}(x+ \\ +\tilde{\xi},t))(\Gamma_i\zeta)(x+\tilde{\xi},t)+\sum_{i=1}^{3}(\sum_{j=1}^{3}I_{j\tau_i}(x+\tilde{\xi},t)(\Gamma_j\zeta)(x+\tilde{\xi},t))\lambda_i]d\tilde{\xi})\}\equiv(\Gamma_0\zeta)(x,t), \\ \tau=x+\tilde{\xi}\in R^3, \end{cases} \quad ...(2.15)$$

at that

$$\begin{cases} \left|D^k\Upsilon_2\right|\le\beta_1,\left|D^kJ\right|\le\beta_2,\forall(x,t)\in D, \\ \Upsilon_{2x_i}=\frac{\partial}{\partial x_i}(\Upsilon_2)=\frac{d_0^{-1}}{\mu\delta_0\sqrt{\pi^3}}\int_0^t\int_{R^3}(\exp(-(\left|\xi\right|^2+\frac{s}{\mu\delta_0})))(\sum_{j=1}^{3}J_{jl_i}(x+2\xi\sqrt{\mu(t-s)},s))\times \\ \times d\xi ds\equiv\Phi_{4,i},(i=\overline{1,3}), \\ \Upsilon_{2x_i^2}=\frac{\partial}{\partial x_i}(\Phi_{4,i}(x,t)),(i=\overline{1,3};l=x+2\xi\sqrt{\mu(t-s)}\in R^3), \\ \frac{1}{\mu\delta_0}\int_0^t exp(-\frac{s}{\mu\delta_0})ds=1-exp(-\frac{t}{\mu\delta_0})\le1,\forall t\in R_+, \\ \Upsilon_{2t}=\Phi_2+\frac{1}{\mu\delta_0\sqrt{\pi^3}}\int_0^t\int_{R^3}(\exp(-(\left|\xi\right|^2+\frac{s}{\mu\delta_0})))\sum_{k=1}^{3}\frac{\xi_k}{\sqrt{t-s}}\sqrt{\mu}J_{l_k}(x+2\xi\sqrt{\mu(t-s)},s)d\xi ds, \\ \left\|\Phi_2\right\|_{1h}=\sup_{R^3}\int_0^\infty h(s)\left|\Phi_2(x,s)\right|ds\le d_0^{-1}\beta_2h_0=\beta_3, \\ \frac{1}{\sqrt{\mu}}\int_0^t(exp(-\frac{s}{\mu\delta_0}))\frac{ds}{\sqrt{t-s}}\le\frac{1}{\sqrt{\mu}}(\int_0^t(exp(-\frac{2(\sqrt{t}-\sqrt{\tau})(\sqrt{t}+\sqrt{\tau})}{\mu\delta_0})\frac{d\tau}{\sqrt{\tau}})^{\frac{1}{2}}(\int_0^t\frac{2d\tau}{2\sqrt{\tau}})^{\frac{1}{2}}\le \\ \le\sqrt{2\delta_0}(\int_0^t(exp(-\frac{2(\sqrt{t}-\sqrt{\tau})\sqrt{t})}{\mu\delta_0})d(-\frac{2}{\mu\delta_0}(\sqrt{t}-\sqrt{\tau})\sqrt{t})^{\frac{1}{2}}\le\sqrt{2\delta_0}, \\ \sup_D\frac{1}{\sqrt{\pi^3}}\int_{R^3}(\exp(-\left|\xi\right|^2))\sum_{k=1}^{3}\left|\xi_k\right|\times\left|J_{l_k}(x+2\xi\sqrt{\mu(t-s)},s)\right|d\xi\le\beta_2\beta_4=\beta_5, \\ \sup_D\frac{1}{\sqrt{\mu\pi^3}}\int_0^t\frac{1}{\sqrt{t-s}}\int_{R^3}(\exp(-(\left|\xi\right|^2+\frac{s}{\mu\delta_0})))\sum_{k=1}^{3}\left|\xi_k\right|\times\left|J_{l_k}\right|d\xi ds\le\beta_5\sqrt{2\delta_0}=\beta_6, \\ \left\|\Upsilon_{2t}\right\|_{1h}\le\beta_3+\beta_6\le\beta_7;\ \left\|\Upsilon_2\right\|_{G_h^1(D_0)}=\sum_{0\le\left|k\le2\right|}\left\|\Upsilon_2\right\|_{C(D)}+\left\|\Upsilon_{2t}\right\|_{1h}\le\beta_8, \\ 0<\beta_i=const,(i=\overline{1,8}) \end{cases} \quad ...(2.16)$$

Similar assessments can be made regarding Υ_1 in $G_h^1(D_0)$, therefore we have $\Upsilon_1+\Upsilon_2=\Upsilon\in G^1(D_0)$.

Since, operator Γ_0 of system (2.15) contains small viscosities δ and $\sqrt{\delta_0}$, then the proof of solvability and the construction of the approximate solution can be realized on the basis of the Banach principle and the Picard's method.

Letting

$$\begin{cases} L_{\Gamma_0} = \bar{k}\sqrt{\delta_0} < 1,\ (0 < \delta_0 < \bar{k}^{-2}, 0 < \bar{k} = const), \\ \Gamma_0 : S_{r_1} \subset S_{r_1}, (S_{r_1}(\zeta_0) = \{\zeta : |\zeta - \zeta_0| \le r_1,\ \forall (x,t) \in D\}), \end{cases} \quad ...(2.17)$$

we obtain for $\zeta(x,t)$ the formula

$$\begin{cases} \zeta = \lim_{n\to\infty} \zeta_{n+1}, \forall (x,t) \in D, \\ \zeta_{n+1} = (\Gamma_0 \zeta_n)(x,t), (n = 0,1,2,3) \end{cases} \quad ...(2.18)$$

we have estimate

$$\|\zeta_{n+1} - \zeta\|_C \le (L_{\Gamma_0})^{n+1} r_1 \xrightarrow[L_{\Gamma_0}<1,(n\to\infty)]{} 0, \quad ...(2.19)$$

here ζ_0 is unitial estimate. Then, taking into account Equation (2.15) we obtain that the function $\theta(x,t)$ is the only one with an estimate

$$\begin{cases} \theta_n = (\Gamma\zeta_n)(x,t), (\zeta_{n+1} = (\Gamma_0 \zeta_n)(x,t), n = 0,1,2,...), \\ \|\theta_n - \theta\|_C \le \mu\delta_0 \|\zeta_n - \zeta\|_C \le \mu\delta_0 L^n_{\Gamma_0} r_1 \xrightarrow[L_{\Gamma_0}<1, n\to\infty]{} 0, \\ \|\zeta_n - \zeta\|_C \le L^n_{\Gamma_0} r_1, (L_{\Gamma_0} = \bar{k}\sqrt{\delta_0} < 1; |\zeta| \le r_2, \forall (x,t) \in D), \\ \|\theta\|_C \le \|\Upsilon\|_C + \mu\delta_0 r_2 \le \beta_9. \end{cases} \quad ...(2.20)$$

It follows from the results obtained that in this case, the pressure becomes known, since the right side Equation (2.10) is a known function.

Further, since Equations (2.14), (2.15) contain $\theta, \theta_{x_i}, (i = 1,2,3), \zeta \in C^{1,0}(D)$, then, taking into account

$$\begin{cases} \theta_{x_i^2} = \Upsilon_{x_i^2} + \dfrac{1}{\sqrt{\pi^3}} \int_0^t \int_{R^3} (\exp(-(|\xi|^2 + \dfrac{s}{\mu\delta_0})))\zeta_{l_i}(x + 2\xi\sqrt{\mu(t-s)}, s) \dfrac{\xi_i}{\sqrt{\mu(t-s)}} d\xi ds, \\ \theta_t = \Upsilon_t + exp(-\dfrac{t}{\mu\delta_0})\zeta(x,t) + \dfrac{1}{\sqrt{\pi^3}} \int_0^t \int_{R^3} (\exp(-(|\xi|^2 + \dfrac{s}{\mu\delta_0})))\sqrt{\mu} \sum_{j=1}^{3} \dfrac{\xi_j}{\sqrt{t-s}} \zeta_{l_j}(x + \\ +2\xi\sqrt{\mu(t-s)}, s) d\xi ds,\ (i = \overline{1,3};\ l = x + 2\xi\sqrt{\mu(t-s)} \in R^3), \end{cases} \quad ...(2.21)$$

moreover, from the estimate Equations (2.14) and (2.21), on the basis of condition (2.16) it follows:

$$\|\theta\|_{G^1_h(D_0)} = \sum_{0 \le |k| \le 2} \|D^k \theta\|_{C(D)} + \|\theta_t(x,t)\|_{1h} \le \beta_{10} = const. \quad ...(2.22)$$

Further, taking into account transformation (2.2), we obtain

$$\begin{cases} \nu_{i,n} = \theta_n \lambda_i + exp(-\dfrac{t}{\mu\delta_0}) J_i(x,t), (i = \overline{1,3};\ n = 0,1,2,...), \\ \|\nu_{i,n} - \nu_i\|_{C(D)} \le \lambda_i \mu\delta_0 L^n_{\Gamma_0} r_1 \xrightarrow[n\to\infty]{} 0. \end{cases} \quad ...(2.23)$$

Then, based on transformation (2.2), conditions (2.22), (2.23) and

$$\begin{cases} \nu_i \in G_h^1(D_0): \ \|\nu_i\|_{G_h^1(D_0)} \le M_0, (0 < M_0 = \beta_{11} + \beta_{12} = const; i = \overline{1,3}), \\ \nu_{it} = \lambda_i \theta_t + exp(-\frac{t}{\mu\delta_0})\{ \ -\frac{1}{\mu\delta_0} J_i + \frac{1}{\sqrt{\pi^3}} \int_{R^3} (exp(-|\xi|^2))\sqrt{\mu} \sum_{j=1}^{3} \frac{\xi_j}{\sqrt{t}} \psi_{il_j}(x + 2\xi\sqrt{\mu t}) d\xi \}, \\ l = x + 2\xi\sqrt{\mu t} \in R^3; \ \sum_{0 \le |k| \le 2} \|D^k \nu_i\|_{C(D)} \le \beta_{11}; \ \|\nu_{it}\|_{1h} \le \beta_{12}, (i = \overline{1,3}), \end{cases}$$

it follows:

$$\|\nu\|_{G_{3,h}^1(D_0)} = \sum_{i=1}^{3} \|\nu_i\|_{G_h^1(D_0)} = \sum_{i=1}^{3} \{ \sum_{0 \le |k| \le 2} \|D^k \nu_i\|_{C(D)} + \sup_{R^3} \int_0^\infty h(t) |\nu_{it}(x,t)| dt \} \le N_0 = const.$$

Theorem 1: Let the Navier-Stokes system (1.1) is defined on the D_0 and with prescribed initial data (1.2), (1.3), and conditions (2.1), (2.8), (2.16) and (2.22). Then there exists a unique solution of the problem (2.12) in $G_h^1(D_0)$. Moreover, taking into account (2.2), there exists solution to problem (1.1), (1.2) and (1.3) in $G_{3,h}^1(D_0)$.

Remark 2: In the case when the functions $\Upsilon_i, (i = 1,2)$ are continuous, the result is valid, if we understand the partial derivatives in the sense of Sobolev (1966). This fact is also one of the significant advantages of the applied method.

3. Conclusion

The main idea of this chapter is that the Navier-Stokes equations (1.1) is reduced to Cauchy problem for inhomogeneous linear equations with the variable coefficients of the heat conduction type, based on the transformation (2.2), taking into account conditions (1.2) and (2.1). The indicated conditions are an important factor for the linearization of Equation (1.1), since condition (2.6) holds when formula (2.2) introduced, i.e., the inertial terms in the Navier-Stokes equations with respect to the new unknown function θ and its derivatives $\theta_{x_i}, (i = 1,2,3)$ are linearized. Further, taking into account (2.2), we also obtain Poisson type equations for pressure of the form Equation (2.9), which modifies the Lipschitz-Landau formula. Therefore, with the exclusion of pressure from Equation (2.7), the linear parabolic problem (2.12) follows, which is reduced to the system of Volterra and Volterra-Abel integral equations of the second kind Equation (2.15), and they simplify the analysis of the original problem in space $G_{3,h}^1(D_0)$.

On the other hand, since the Navier-Stokes equations with certain initial conditions were studied in papers (Omurov, 2019; 2021) in $G_3^1(D_1 = R^3 \times (0,T_0))$ and $W_3^0(D)$ (see sector 1), and in this work this equation is studied with conditions (1.3), (1.20) in $G_3^1(D_0)$ (see sector 2). Therefore, it can be considered that the Navier-Stokes equation is studied in the full sense with the Cauchy conditions. Note that in the future, space $G_3^1(D_0)$ can be used for the Navier-Stokes problem in a bounded domain, when D_0 is bounded.

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Cite this article as: Taalaibek D. Omurov (2023). A Solution of the Navier-Stokes Problem for an Incompressible Fluid with Cauchy Condition. *International Journal of Pure and Applied Mathematics Research*, 3(2), 33-47. doi: 10.51483/IJPAMR.3.2.2023.33-47.