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Asymptotic Synchronization of Nonlinear Functional Neutral Delay Difference Equations with Variable Coefficients

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Abstract

We mathematically modelled the process of asymptotic synchronization in all-to-all coupled structures using a formulated discrete protocol with time delays. We utilized the oscillatory nature of synchronization to transform the discrete protocol onto quantinuum dynamics of a known class of first order nonlinear neutral delay difference equations (NDDE) with variable coefficients. We applied some mathematical inequality techniques to obtain bounded solutions of the NDDE. We utilized the known oscillatory property of all neutral delay difference equations to classify the bound solutions as asymptotically synchronizing over a given time domain projected through the initial and boundary conditions. The solutions obtained are in the form of converging sequences. Some illustrative examples were provided to validate the main results.

Keywords: *Asymptotic synchronization, Mathematical inequality, Variable coefficients*

1. Introduction

Asymptotic synchronization in coupled dynamical agents can be presented mathematically through the following dynamic discrete protocol:

$$u_i(t) = \sum_{\vartheta_i \in N_i} \alpha_{ij}\left(x_j\left(t-\tau_{ij}(t)\right) - \left(x_i\left(t-\tau_{ij}(t)\right)\right)\right), \qquad ...(1)$$

where, α_{ij} is the coupling strength in coupled structures, and $\tau_{ij}(t)$ is the time varying delay.

In application, the discrete protocol (1) models the oscillatory phenomena in a wide class of coupled dynamical agents such as oscillators.

Because of the oscillatory nature of synchronization, we present Equation (1) with quantinuum dynamics of a known class of first order nonlinear neutral delay difference equations with variable coefficients of the form,

$$\Delta(\alpha(t)\,(A(t)x(t) + \beta(t)x(t-\tau_1))) + \varphi(t)f(x(t-\tau_2)) = 0,\ t \geq t_0; \qquad ...(2)$$

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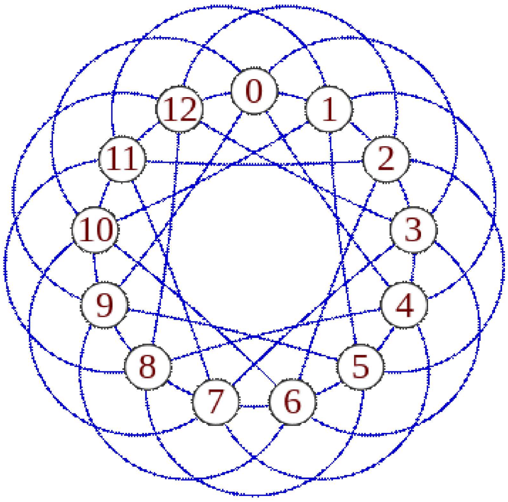

Figure 1: The All-to-All Coupling of Identical Oscillators

where, $\Delta x(t) = x(t+1) - x(t)$ with the forward difference operator Δ, and $\{A(t)\}$, $\{\alpha(t)\}$, $\{\beta(t)\}$ and $\{\varphi(t)\}$ are sequences of positive real numbers defined on $T(t_0) = \{t_0, t_0+1, ...\}$, (Wang *et al.*, 2006).

We study the dynamics of Equation (2) on a strongly regular network provided in Figure (1) below. In graph theory, a graph $\mathcal{G}(V, E)$ is described as being strongly regular (Andries and Hendrik, 2022) if there are time-varying integers β and such that

- Every two adjacent vertices have β moving common neighbours.
- Every two non-adjacent vertices have φ moving common neighbours.

For n number of oscillators, the integer values and are varying in time (by varying the looping) to satisfy the following cases

- $\beta = n - 1$, $\varphi = 0$,
- $\beta = \varphi$
- $\beta < \varphi$

In the coupled dynamical networks, $\alpha(t)$ represent the coupling strength, and $A(t)$ is the synchronization parameter. Whenever the function f is convergent, then equation (2) asymptotically synchronizes.

Some sufficient conditions in relation to the coefficients of Equation (2) were applied in form of inequalities to obtain the asymptotic synchronization of all solutions of Equation (2). The main results were obtained using four inequality techniques (Bazighifan, 2021; Grace, 2020).

2. Background

Equation of type (2) with $A(t) = 1$, has been studied in literature for the oscillatory behavior of difference equations. However, the significance in application of the case with $A(t) \neq 1$, has not been presented in literature. Synchronization remain a process endowed with oscillations (Almarri *et al.*, 2022). The purpose of this study is to present the properties necessary for the process of asymptotic synchronization of (2).

***Assumptions*:** The following conditions are assumed to hold.

(i) $\tau_1 > 0$ and $\tau_2 > 0$;

(ii) There exist constants A_0, β_0 and β_1 such that $A(t) \leq A_0$ and $\beta_0 \leq \beta(t) \leq \beta_1 < \infty$;

(iii) $f: R \to R$ is a continuous function satisfies $vf(v) > 0$ for $v \neq 0$;

(iv) There exists a positive constant k_0 such that $\dfrac{f(v)}{v} \geq k_0 > 0$

A solution of Equation (2) on $T(t_0) = \{t_0, t_0 + 1, \ldots\}$, is considered as a real sequence $\{x(t)\}$ which is defined on $t \geq t_0 - t^*$ (where, $t^* > \max\{\tau_1, \tau_2\}$ is a chosen positive integer) which satisfies (2) for $t \in T(t_0)$. A solution $\{x(t)\}$ of (2) on $T(t_0)$ is said to asymptotically synchronize if for every positive integers $T(t_0) > t_0$ there exists $t \geq T_0$ such that $x(t)x(t+1) \leq 0$, otherwise $\{x(t)\}$ is said to be non-synchrony (Alzabut *et al.*, 2021).

3. Governing Auxiliary Lemmas

In this section, some useful lemmas are given which governed in the study of the asymptotic synchronization of Equation (2).

Lemma 3.1. (Gyori and Ladas, 1991)

Assuming that τ_2 is a positive integer and $\{\varphi(t)\}$ is a sequence of positive real numbers, then the difference inequality (3) has an asymptotic positive solution whenever the difference Equation (4) has an asymptotic positive solution.

$$\Delta x(t) + \varphi(t)x(t - \tau_2) \leq 0, \qquad t \geq t_0 \qquad \text{...(3)}$$

Lemma 3.2.

Letting

$$\lim_{t\to\infty} \sup \sum_{s=t}^{t+\tau_2} \varphi(\alpha) > 0 \qquad \text{...(4)}$$

and if $\{x(t)\}$ is an asymptotic positive solution of the delay difference Equation (2), then,

$$\lim_{t\to\infty} \inf \frac{x(t-\tau_2)}{x(t)} < \infty \qquad \text{...(5)}$$

Proof:

Assuming that there exists a sequence $\{t_k\}$ of integers and a constant $D > 0$ such that $t_k \to \infty$ as $k \to \infty$ and

$$\lim_{t\to\infty} \sup \sum_{s=t_k}^{t_k+\tau_2} \varphi(s) \geq D, \qquad k = 1, 2, 3, \ldots. \qquad \text{...(6)}$$

Then there exists $\theta \in \{t_k, t_{k+1}, \ldots, t_{k+\tau_2}\}$ for every k such that

$$\sum_{s=t_k}^{\theta_k} \varphi(s) \geq \frac{D}{2} \qquad \text{...(7)}$$

And

$$\sum_{s=\theta_k}^{t_k+\tau_2} \varphi(s) \geq \frac{D}{2} \qquad \text{...(8)}$$

Summing the Equation (5) for t_k to μ_k and μ_k to $t_k + \tau_2$, we find

$$(\mu_k + 1) - x(t_k) + \sum_{s=t_k}^{\mu_k} \varphi(s)x(s - \tau_2) = 0 \qquad \text{...(9)}$$

and

$$x(t_k - \tau_2 + 1) - x(t_k) + \sum_{s=\mu_k}^{t_k+\tau_k} \varphi(s)x(s - \tau_2) = 0 \qquad \backslash \qquad \text{...(10)}$$

By omitting the first terms in Equation (10) and (14), and by using the decreasing nature of $\{x(t)\}$ and Equation (7) and (8), we obtain

$$-x(t_k)+\frac{D}{2}x(\mu_k-\tau_2)\le 0 \qquad ...(11)$$

and

$$-x(t_k)+\frac{D}{2}x(t_k)\le 0 \qquad ...(12)$$

or

$$\frac{x(\theta_k-\tau_2)}{x(\theta_k)}\le\left(\frac{D}{2}\right)^2 \qquad ...(13)$$

Lemma 3.3.

Assuming that $\tau_2>\tau_1$, $\alpha(t)\equiv 1$ and

$$\lim_{t\to\infty} sup \sum_{s=t}^{t+\tau_2-\tau_1}\varphi(s)>0 \qquad ...(14)$$

Let $\{x(t)\}$ be an asymptotic positive solution of the Equation (1). Set

$$m(t)=A(t)x(t)+\beta(t)x(t-\tau_1) \qquad ...(15)$$

Then,

$$\lim_{t\to\infty} sup \frac{m(t-\tau_2+\tau_1)}{m(t)}<\infty \qquad ...(16)$$

Proof:

Considering Equations (2) and (16), we have $m(t)>0$ asymptotically and decreasing. From Equation (15) it yields,

$$m(t+\tau_1)\ge A(t+\tau_1)x(t+\tau_1)+\beta_0 x(t) \qquad ...(17)$$

and

$$\beta(t)x(t-\tau_1)=m(t)-A(t)x(t)\ge m(t)-A_0x(t) \qquad ...(18)$$

Since $\{m(t)\}$ is decreasing, we have $\{m(t)\}>m(t+\tau_1)\}$ which implies with Equation (17) that $m(t)\ge\beta_0 x(t)$ or

$$x(t)\le\frac{m(t)}{\beta_0}$$

Using the above inequality in (18), we obtain

$$\beta(t)x(t-)\ge m(t)-\frac{A_0 m(t)}{\beta_0}$$

or $\beta_1 x(t-\tau_1)\ge\dfrac{m(t)(\beta_0-A_0)}{\beta_0}$,

or $\beta_1 x(t-\tau_1)\ge\dfrac{m(t)(\beta_0-A_0)}{\beta_0\beta_1}$. ...(19)

Hence

$$x(t-\tau_2)\ge\frac{m(t)(\beta_0-A_0)}{\beta_0\beta_1}m(t+\tau_1-\tau_2) \qquad ...(20)$$

From Equation (2) and Equation (20), we get

$$\Delta m(t)+\frac{k_0(\beta_0-A_0)}{\beta_1\beta_0}\alpha(t)m(t+\tau_1-\tau_2)\le 0 \quad ...(21)$$

Using Lemma 2.1, we find that the Equation (22) below has an asymptotic positive solution as well.

$$\Delta m(t)+\frac{k_0(\beta_0-A_0)}{\beta_0\beta_1}\varphi(t)m(t+\tau_1-\tau_2)=0 \quad ...(22)$$

As a result, using Lemma 3.2 and Equation (14), we have

$$\lim_{t\to\infty} inf\,\frac{m(t+\tau_1-\tau_2)}{m(t)}<\infty \quad ...(23)$$

Which is the desired result. The proof is complete.

Lemma 3.4.

Assume that $\tau_2>\tau_1$, and $\alpha(t)\equiv 1$. If Equation (2) has an asymptotically positive solution, then

$$\sum_{s=t}^{t+\tau_2-\tau_1}\varphi(s)\le\frac{\beta_0\beta_1}{k_0(\beta_0-A_0)} \quad ...(24)$$

for all sufficiently large t.

Proof:

Proceeding as in the proof of Lemma 3.3, the inequality (21) is obtained. Summing Equation (21) from t to $t+\tau_2-\tau_1$, we get

$$m(t-\tau_1+\tau_2+1)-m(t)+\frac{k_0(\beta_0-A_0)}{\beta_1\beta_0}\sum_{s=t}^{t+\tau_2-\tau_1}\varphi(s)m(s+\tau_1-\tau_2)\le 0 \quad ...(25)$$

Using the decreasing nature of $\{m(t)\}$ gives the result

$$m(t-\tau_1+1)-m(t)+\frac{k_0(\beta_0-A_0)}{\beta_1\beta_0}m(t)\sum_{s=t}^{t+\tau_2-\tau_1}\varphi(s)\le 0 \quad ...(26)$$

Then,

$$m(t-\tau_1+\tau_2+1)-m(t)+\left(\frac{k_0(\beta_0-A_0)}{\beta_1\beta_0}m(t)\sum_{s=t}^{t+\tau_2-\tau_1}\varphi(s)-1\right)m(t)\le 0 \quad ...(27)$$

For sufficiently large t, we obtain

$$\sum_{s=t}^{t+\tau_2-\tau_1}\varphi(s)\le\frac{\beta_0\beta_1}{k_0(\beta_0-\beta_0)} \quad ...(28)$$

which is the desired result. The proof is completed.

4. Asymptotic Synchronization of Solutions

Theorem 4.1.

Assume that $\tau_2>\tau_1$, and $\alpha(t)\equiv 1$ and Equation (15) holds. If

$$\sum_{t=t_0}^{\infty}\varphi(t)ln\left(\frac{ek_0(\beta_0-A_0)}{\beta_1\beta_0}\sum_{s=t}^{t+\tau_2-\tau_1}\varphi(s)\right)=\infty \qquad ...(29)$$

Then every solution of Equation (2) asymptotically synchronizes.

Proof:

Assume the contrary. To generalize assume that $\{x(t)\}$ is an asymptotic positive solution of Equation (2). Set $m(t)$ as in Equation (14). Then $\{m(t)\}$ is asymptotic positive and decreasing. Also $\{x(t)\}$ satisfies the inequality Equation (15). That is,

$$\Delta m(t)+\frac{k_0(\beta_0-A_0)}{\beta_1\beta_0}\varphi(t)m(t+\tau_1-\tau_2)\le 0 \qquad ...(30)$$

Define the sequence $\{\vartheta(t)\}$ as

$$\vartheta(t)=\frac{-\Delta m(t)}{m(t)} \qquad ...(31)$$

Then $\{\vartheta(t)\}$ is asymptotically non-negative. So, there exists $t_1\ge t_0$ with $m(t_1)>0$. It can be shown that

$$\Delta m(t)\le m(t_1)exp\left(-\sum_{s=t_1}^{t-1}\varphi(s)\right) \qquad ...(32)$$

Moreover, $\{\vartheta(t)\}$ satisfies

$$\vartheta(t)\ge\frac{ek_0(\beta_0-A_0)}{\beta_1\beta_0}\varphi(t)e \qquad ...(33)$$

$$\vartheta(t)\ge\frac{ek_0(\beta_0-A_0)}{\beta_1\beta_0}\varphi(t)exp\left(\sum_{s=t+\tau_1-\tau_2}^{t-1}u(s)\right) \qquad ...(34)$$

By using the inequality

$$e^{ax}\ge x+\frac{ln(e\alpha)}{\alpha}, \qquad x,\alpha\ge 0 \qquad ...(35)$$

we have from (28),

$$\vartheta(t)\ge\frac{ek_0(\beta_0-A_0)}{\beta_1\beta_0}\varphi(t)exp\left(\frac{Q(t)}{Q(t)}\sum_{S=t+\tau_1-\tau_2}^{t-1}u(s)\right)$$

$$\ge\frac{ek_0(\beta_0-A_0)}{\beta_1\beta_0}\varphi(t)\left(\frac{1}{Q(t)}\sum_{s=t+\tau_1-\tau_2}^{t-1}\vartheta(s)+\frac{ln(eQ(t))}{Q(t)}\right) \qquad ...(36)$$

where,

$$Q(t)=\frac{ek_0(\beta_0-A_0)}{\beta_1\beta_0}\sum_{S=t+1}^{t-1}(s) \qquad ...(37)$$

Therefore,

$$\vartheta(t)=\sum_{S=t+1}^{t+\tau_2-\tau_1}\vartheta(s)-\varphi(t)\sum_{s=t+\tau_1-\tau_2}^{t-1}\vartheta(s)\ge\varphi(t)ln\left[\frac{ek_0(\beta_0-A_0)}{\beta_1\beta_0}\sum_{S=t+1}^{t+\tau_2-\tau_1}\varphi(s)\right] \qquad ...(38)$$

Hence for $\mu > T + \tau_2 - \tau_1$

$$\sum_{t+1}^{\mu-1} \vartheta(t)\left[\sum_{s=t+1}^{t+\tau_2-\tau_1} \varphi(s)\right] - \sum_{T+1}^{\mu-1} \varphi(t)\left(\sum_{s=t+\tau_2-\tau_1}^{t-1} \vartheta(s)\right)$$

$$\geq \sum_{T+1}^{\mu-1} \varphi(t) ln\left(\frac{ek_0(\beta_0 - A_0)}{\beta_1\beta_0} \sum_{S=t+1}^{t+\tau_2-\tau_1} \varphi(s)\right) \quad ...(39)$$

By interchanging the order of summation, we have

$$\sum_{t+1}^{\mu-1} \varphi(t) \sum_{S=t+\tau_1-\tau_2}^{t-1} \vartheta(s) \sum_{t=T}^{\mu+\tau_1-\tau_2-1} \vartheta(t) \sum_{S=t+1}^{t-\tau_1+\tau_2} \varphi(t) \quad ...(40)$$

Combining Equations (47) and (48) leads to

$$\sum_{t+1}^{\mu-1} \vartheta(t) \sum_{S=t+1}^{t+\tau_2-\tau_1} \varphi(s) \geq \sum_{t+1}^{\mu-1} \varphi(t)\ ln\left(\frac{ek_0(\beta_0 - A_0)}{\beta_1\beta_0} \sum_{s=t+1}^{t+\tau_2-\tau_1} \varphi(s)\right) \quad ...(41)$$

Using Equation (27) of Lemma 2.4 in Equation (39), we obtain

$$\sum_{t=\mu-\tau_2+\tau_1}^{\mu-1} \vartheta(t) \geq \frac{ek_0(\beta_0 - A_0)}{\beta_1\beta_0} \sum_{t=T}^{\xi-1} \varphi(t)\ ln\left(\frac{ek_0(\beta_0 - A_0)}{\beta_1\beta_0} \sum_{s=t+1}^{t+\tau_2-\tau_1} \varphi(s)\right) \quad ...(42)$$

or

$$ln\frac{m(\mu+\tau_2-\tau_1)}{m(t)} \geq \frac{k_0(\beta_0 - A_0)}{\beta_1\beta_0} \sum_{t=T}^{\mu-1} \varphi(t)\ ln\left(\frac{k_0(\beta_0 - A_0)}{\beta_1\beta_0} \sum_{s=t+1}^{t+\tau_2-\tau_1} \varphi(s)\right) \quad ...(43)$$

This result along with condition (34) leads to

$$\lim_{t\to\infty} \frac{m(t+\tau_2-\tau_1)}{m(t)} = \infty \quad ...(44)$$

which contradicts Equation (41) and completes the proof.

Theorem 4.2.

Assume that $\tau_2 > \tau_1$, and $\Delta\alpha(t) > 0$.

If

$$0 < C \leq \lim_{t\to\infty} inf \sum_{S=t}^{t+\tau_2-\tau_1} \frac{\varphi(s)}{\alpha(s+\tau_2-\tau_1)} \quad ...(45)$$

and

$$\sum_{S=t}^{t+\tau_2-\tau_1} \frac{\varphi(t)}{\alpha(t+\tau_2-\tau_1)}\ ln\left(\frac{ek_0(\beta_0 - A_0)}{\beta_1\beta_0} \sum_{S=t}^{t+\tau_2-\tau_1} \frac{\varphi(s)}{\alpha(t+\tau_2-\tau_1)}\right) = \infty \quad ...(46)$$

then every solution of (2) asymptotically synchronizes.

Proof:

We assume that $\{x(t)\}$ is an eventually positive solution of (2). Set $m(t)$ as in (27). Then $\{m(t)\}$ is positive and decreasing. Proceeding as in the proof of Lemma 3.3, we get

$$\Delta(\alpha(t)m(t)) + \frac{k_0(\beta_0 - A_0)}{\beta_1\beta_0} \varphi(t) m(t+\tau_2-\tau_1) \leq 0 \quad ...(47)$$

Set

Using this in (45), we get

$h(t) = \alpha(t)m(t)$

$$\Delta h(t) + \frac{k_0(\beta_0 - A_0)}{\beta_1\beta_0}\frac{\varphi(t)}{(t+\tau_2-\tau_1)}h(t+\tau_2-\tau_1) \le 0 \qquad \text{...(48)}$$

Set

$$\lambda(t) = -\frac{\Delta h(t)}{h(t)}$$

Then $\lambda(t) > 0$ asymptotically and $(\lambda(t))$ satisfies the inequality

$$\Delta\lambda(t) \ge \frac{k_0(\beta_0 - A_0)}{\beta_1\beta_0}\frac{\varphi(t)}{\alpha(t+\tau_2-\tau_1)}exp\left(\sum_{S=t+\tau_2-\tau_1}^{t-1}\lambda(s)\right) \qquad \text{...(49)}$$

Applying the inequality (58) to (57) yields

$$e^{ax} \ge x + \frac{ln(e\alpha)}{\alpha}, \qquad x, \alpha > 0 \qquad \text{...(50)}$$

$$\Delta\lambda(t) \ge \frac{k_0(\beta_0 - A_0)}{\beta_1\beta_0}\left(\frac{\varphi(t)}{\alpha(t+\tau_2-\tau_1)}\right)exp\left(\frac{d(t)}{d(t)}\sum_{S=t+\tau_2-\tau_1}^{t-1}\lambda(s)\right)$$

$$\ge \left(\frac{k_0(\beta_0 - A_0)}{\beta_1\beta_0}\right)\left(\frac{\varphi(t)}{\alpha(t+\tau_2-\tau_1)}\right)\left(\frac{1}{D(t)}\sum_{s=t+\tau_2-\tau_1}^{t-1}\lambda(s) + \frac{ln(eD(t))}{D(t)}\right) \qquad \text{...(51)}$$

where

$$d(t) = \frac{k_0(\beta_0 - A_0)}{\beta_1\beta_0}\sum_{s=t+1}^{t+\tau_2-\tau_1}\frac{\varphi(t)}{\alpha(t+\tau_2-\tau_1)} \qquad \text{...(52)}$$

Therefore,

$$\lambda(t)\sum_{S=t+1}^{t+\tau_2-\tau_1}\frac{\varphi(t)}{\alpha(t+\tau_2-\tau_1)} - \frac{\varphi(t)}{\alpha(t+\tau_2-\tau_1)}\sum_{S=t+\tau_2-\tau_1}^{t-1}\lambda(s)$$

$$\ge \frac{\varphi(t)}{\alpha(t+\tau_2-\tau_1)}ln\left(\frac{ek_0(\beta_0 - A_0)}{\beta_1\beta_0}\sum_{s=t+1}^{t+\tau_2-\tau_1}\frac{\varphi(s)}{\alpha(s+\tau_1-\tau_2)}\right) \qquad \text{...(53)}$$

Hence, for $\mu > t + \tau_2 - \tau_1$,

$$\sum_{t=T_1}^{\mu-1}\lambda(t)\sum_{S=t+1}^{t+\tau_2-\tau_1}\frac{\varphi(t)}{\alpha(t+\tau_2-\tau_1)} - \sum_{t=T_1}^{\mu-1}\frac{\varphi(t)}{\alpha(t+\tau_2-\tau_1)}\sum_{S=t+\tau_2-\tau_1}^{t-1}\lambda(s)$$

$$\ge \sum_{t=T_1}^{\eta-1}\frac{\varphi(t)}{\alpha(t+\tau_1-\tau_2)}\ ln\left(\frac{ek_0(\beta_0 - A_0)}{\beta_1\beta_0}\sum_{s=t+1}^{t+\tau_2-\tau_1}\frac{\varphi(s)}{\alpha(t+\tau_1-\tau_2)}\right) \qquad \text{...(54)}$$

By interchanging the order of summation, we have

$$\sum_{t=T_1}^{\mu-1}\frac{\varphi(t)}{\alpha(t+\tau_2-\tau_1)}\left(\sum_{s=t+\tau_2-\tau_1}^{t-1}\lambda(s)\right)\geq\sum_{t=T_1}^{t+\tau_2-\tau_1-1}\lambda(t)\sum_{s=t+1}^{t+\tau_2-\tau_1}\frac{\varphi(s)}{\alpha(t+\tau_2-\tau_1)} \qquad ...(55)$$

From (46) and (47), we have

$$\sum_{t=t+\tau_2-\tau_1}^{\mu-1}\lambda(t)\sum_{s=t+1}^{\mu+\tau_2-\tau_1}\frac{\varphi(t)}{\alpha(t+\tau_2-\tau_1)}$$

$$\geq\sum_{s=T}^{\mu-1}\frac{\varphi(t)}{\alpha(t+\tau_1-\tau_2)}\,ln\left(\frac{ek_0(\beta_0-A_0)}{\beta_1\beta_0}\sum_{S=t+1}^{t+\tau_2-\tau_1}\frac{\varphi(s)}{\alpha(s+\tau_1-\tau_2)}\right) \qquad ...(56)$$

Using (38) in (48), it follows that

$$\sum_{t=t+\tau_2-\tau_1}^{\eta-1}\lambda(t)\geq\frac{ek_0(\beta_0-A_0)}{\beta_1\beta_0}\sum_{t=N_1}^{t+\tau_2-\tau_1-1}\frac{\varphi(t)}{\alpha(t+\tau_1-\tau_2)}\,ln\left(\frac{ek_0(\beta_0-A_0)}{\beta_1\beta_0}\sum_{S=t+1}^{t+\tau_2-\tau_1}\frac{\varphi(s)}{\alpha(t+\tau_2-\tau_1)}\right) \qquad ...(57)$$

or

$$ln\frac{h(\mu+\tau_2-\tau_1)}{h(\mu)}\geq\frac{ek_0(\beta_0-A_0)}{\beta_1\beta_0}\sum_{t=N_1}^{t+\tau_2-\tau_1-1}\frac{\varphi(t)}{\alpha(t+\tau_2-\tau_1)}\,ln\left(\frac{ek_0(\beta_0-A_0)}{\beta_1\beta_0}\sum_{S=t+1}^{t+\tau_2-\tau_1}\frac{\varphi(s)}{\alpha(t+\tau_1-\tau_2)}\right) \qquad ...(58)$$

From (47) and (49), we have

$$\lim_{n\to\infty}\frac{h(t+\tau_2-\tau_1)}{h(\mu)}=\infty \qquad ...(59)$$

On the other hand, from condition (45), there exists a sequence $\{t_k\}$ of integers, $t_k\to\infty$ as $k\to\infty$, and there exists $t_k^*\in\{t_k,t_k+1,\ldots,t_k+\tau_2-\tau_1\}$ for every k such that

$$\sum_{t=T_k}^{T_k^*}\frac{\varphi(t)}{\alpha(t+\tau_2-\tau_1)}\geq\frac{Q}{2}\ and\ \sum_{S=T_k}^{T_k+\tau_2-\tau_1}\frac{\varphi(s)}{\alpha(s+\tau_1-\tau_2)}\geq\frac{Q}{2} \qquad ...(60)$$

Summing both sides of (36) from t_k to t_k^* and t_k^* to $t_k+\tau_2-\tau_1$ we have

$$h(t_k^*+1)-h(t_k)+\frac{k_0(\beta_0-A_0)}{\beta_1\beta_0}\sum_{s=t_k}^{T_k^*}\frac{\varphi(s)h(s+\tau_1-\tau_2)}{\alpha(s+\tau_2-\tau_1)}\leq 0 \qquad ...(61)$$

and

$$h(t_k+\tau_2-\tau_1+1)-y(t_k^*)+\frac{k_0(\beta_0-A_0)}{\beta_1\beta_0}\sum_{S=t_k}^{t_k+\tau_2-\tau_1}\frac{\varphi(s)}{\alpha(s+\tau_1-\tau_2)}h(s+\tau_1-\tau_2)\leq 0 \qquad ...(62)$$

Using the decreasing nature of $\{h(n)\}$ and from (60), (61) and (62), we get

$$-h(t_k)+\frac{k_0(\beta_0-A_0)Q}{2\beta_1\beta_0}h(t_k^*+\tau_1-\tau_2)\leq 0 \qquad ...(63)$$

and

$$-h(t_k^*)+\frac{k_0(\beta_0-A_0)Q}{2\beta_1\beta_0}h(t_k)\leq 0 \qquad ...(64)$$

This implies eventually,

$$\frac{h\left(t_k^* + \tau_1 - \tau_2\right)}{h\left(t_k^*\right)} \le \left(\frac{2\beta_1\beta_0}{k_0\left(\beta_0 - A_0\right)Q}\right)^2 \qquad ...(65)$$

which is a contradiction with Equation (59). The proof is complete.

5. Some Examples

In this section we give some examples to illustrate our results.

Example 5.1.

Consider the following first order neutral delay difference equation

$$\Delta\left[\frac{1}{t}x(t) + \left(4 + \frac{1}{t}\right)x(t-5)\right] + \frac{2}{3}\left(3 + \frac{1}{t} + \frac{1}{t+3}\right)x(t-7)\left(2 + x^2(t-7)\right) = 0$$

$$t \ge 7 \qquad ...(66)$$

we have

$$\alpha(t) \equiv 1,\ A(t) = \frac{1}{t},\ \varphi(t) = \frac{2}{3}\left(3 + \frac{1}{t} + \frac{1}{t+3}\right),\ \tau_2 = 7,\ \tau_1 = 5,\ \beta(t) = 4 + \frac{1}{t}$$

$$f\left(x(t-4)\right) = x(t-7)\left(2 + x^2(t-7)\right)$$

We can easily see that $k_0 = 2$ and $1 < 5 \le \beta(t) \le \frac{27}{8} < \infty$.

Now,

$$\sum_{t=5}^{\infty}\varphi(t)\, ln\left(\frac{ek_0\left(\beta_0 - A_0\right)}{\beta_1\beta_0}\sum_{S=t+1}^{t+\tau_2-\tau_1}\varphi(s)\right)$$

$$= \sum_{t=5}^{\infty}\frac{2}{3}\left(2 + \frac{1}{t} + \frac{1}{t+3}\right) ln\left(\frac{8e}{27}\left(5 + \frac{1}{t+1} + \frac{1}{t+2} + \frac{1}{t+3}\right)\right) > \frac{2}{3}\sum_{t=5}^{\infty}\left(2 + \frac{1}{t} + \frac{1}{t+1}\right)$$

$$= \infty \qquad ...(67)$$

By Theorem 4.1 every solution of Equation (66) asymptotically synchronizes. One such solution of Equation (67) is $x(t) = (-1)^t$.

Example 5.2.

We consider the first order delay differential equation given as

$$\Delta\left[t\left(\frac{1}{t}x(t) + \left(3 - \frac{1}{t}\right)x(t-2)\right)\right] + \frac{3}{5}(3t-2)t(t-5)x\left(3 + x^2(t-5)\right) = 0 \qquad ...(68)$$

In this case, $\alpha(t) \equiv 1,\ A(t) = \frac{1}{t},\ \varphi(t) = \frac{2}{3}(3t-2),\ \tau_2 = 5,\ \tau_1 = 1,\ \beta(t) = 3 - \frac{1}{t}$,

$f\left(x(t-5)\right) = x(t-5)\left(3 + x^2(t-5)\right)$. Also, $k_0 = 3$ and $1 < \frac{7}{5} \le \beta(t) \le 3 < \infty\ A(t) = \frac{1}{5}$ and

$$\lim_{t\to\infty} inf \sum_{S=t}^{t+\tau_2-\tau_1} \frac{\varphi(s)}{\alpha(s+\tau_2-\tau_1)}$$

$$=\frac{3}{5}\lim_{t\to\infty} inf \sum_{S=t}^{t+3} \frac{3S-1}{S-3}$$

$$=\frac{3}{5}\lim_{t\to\infty} inf \sum_{S=t}^{t+3} 2+\frac{5}{S-3}=6>0 \quad ...(69)$$

6. Conclusion

In this study we established the discrete consensus dynamics of the networked heterogeneous systems for delay-induced framework of oscillators coupled all-to-all with non-identical time-varying lags. Because of the oscillatory nature of the process synchronization, the discrete consensus protocol (1) was transformed to the continuum dynamics (2) through the coupling strength, Equation (2) was solved using some inequality techniques to obtain solutions in the form of mathematical sequences $\{x(t)\}$. These solutions imply that synchronization in coupled dynamical structures converges.

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