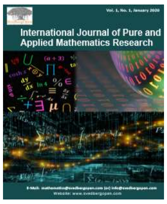
https://doi.org/10.51483/IJPAMR.3.2.2023.1-19 ISSN: 2789-9160

International Journal of Pure and Applied Mathematics Research

Publisher's Home Page: https://www.svedbergopen.com/



Research Paper Open Access

Multidimensional Complex Vectors

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Article Info

Volume 3, Issue 2, October 2023
Received : 21 November 2022
Accepted : 19 August 2023
Published : 05 October 2023
doi: 10.51483/IJPAMR.3.2.2023.1-19

Abstract

A normal or pure vector **r** is one whose components are scalars multiplied by the unit basis vectors. It is shown that factors of the unit bases vectors, to represent a complex vector, in its simplest form become complex numbers, $(a_i + jb_i)$, for $j = \sqrt{-1}$. Thus, each factor of each axis can have a real part and an imaginary part. It is known that both the real and imaginary parts of a complex number are imagined as lines perpendicular to each other. From this, an imaginary and perpendicular axis is conceived as created by each real axis for defining N-Dimensional complex vectors (or pure, for $j = 0$). Multidimensional Dot and Cross product of two complex vectors were achieved. Similarly, Multidimensional Dot and Cross Triple Product of three complex vectors were obtained. In this way, new multidimensional properties of pure and complex vectors definitions were found.

Keywords: *Multidimensional complex vectors, Multidimensional dot and cross products of two complex vectors, Multidimensional dot and cross products of three complex vectors*

1. Introduction

As has been known for a long time, complex numbers appear, for example, as solution of system of equations to find the intersection of curves or lines that do not actually intersect. The obtained roots that qualify this non-existent cutoff point are what has been called complex numbers. In following Section 2, we used the strategy of solving a system of equations of lines that do not intersect, such as the case of the two-dimensional parabola and the linear equation of its directrix located in its same plane, finding its complex roots in a natural vector form. The analysis of the result lead to the requirement of a four-dimensional space to be represented, with two real axes and two imaginary ones. The problem was then posed within a three-dimensional space, and its complex roots required six dimensions, confirming the 2N-dimensional structure obtained earlier. With these results, in Sections 3 and 4 a formal definition of complex vectors was given. In Section 5 the multidimensional dot and cross products of pure and complex vectors were developed. In Section 6, a discussion of these results is given. In Sections 7 and 8 were achieved the multidimensional Cross and Dot vector triple products. Section 9 has a summary of complex vector features and Section 10 our conclusions.

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2. Cut of a Parabola with its Directrix

2.1 Two-Dimensional Cut of a Parabola with its Directrix

As it is known, a parabola is the locus of the points equidistant from a point called focus and a straight line called directrix also located in the plane of the curve, see Figure 1. By definition they never intersect. The distances denoted by d from any point $P(x, y)$ of the curve to the focus $F(g, h)$, is $d^2 = (x - g)^2 + (y - h)^2$ and to the directrix, $d^2 = \frac{|m(x-x_0) - l(y-y_0)|^2}{l^2+m^2}$ respectively; where the values of l and m define the direction of the directrix, which passes through the point $P(x_0, y_0)$, also a point of the axis of the parabola. Equating the two distances, we can obtain another general equation of the Parabola, which depends on the focus $F(g, h)$, the point of intersection of the directrix and the axis of symmetry, and the values of l and m, which define the orientation of the directrix. On the other hand, any point on the directrix must satisfy: $\frac{x-x_0}{l} = \frac{y-y_0}{m}$, or: $y = \frac{m}{l}(x-x_0)+y_0$ and $x = \frac{l}{m}(y-y_0)+x_0$; thus, its equation also corresponds to: $l(y - y_0) - m(x - x_0) = 0$. The values of l and m defining the directrix can be any pair of values that are kept proportional to each other. Solving the system of equations coming from establishing the equality of the distances from a point $P(x, y)$ in the curve to the Focus F and to its directrix, $d = \overline{PF} = \overline{PP'}$, the values (x, y) of the "cut-off point" of the parabola and its directrix are obtained:

$$(x-g)^2 + (y-h)^2 = \frac{|l(y-y_0) - m(x-x_0)|^2}{l^2+m^2} = \frac{\begin{vmatrix} l & m \\ x-x_0 & y-y_0 \end{vmatrix}^2}{l^2+m^2}$$

$$\left\{ \begin{matrix} (x-g)^2 + (y-h)^2 - \frac{\begin{vmatrix} l & m \\ x-x_0 & y-y_0 \end{vmatrix}^2}{l^2+m^2} = 0 \\ l(y-y_0) - m(x-x_0) = 0 \end{matrix} \right\} \quad \text{...(1)}$$

Substituting the y value by the expression $y = \frac{m}{l}(x-x_0)+y_0$, coming from the equation of the directrix; multiplying by l^2 and making $p_x = lg$, and $q_x = mx_0 - l(y_0 - h)$, we have:

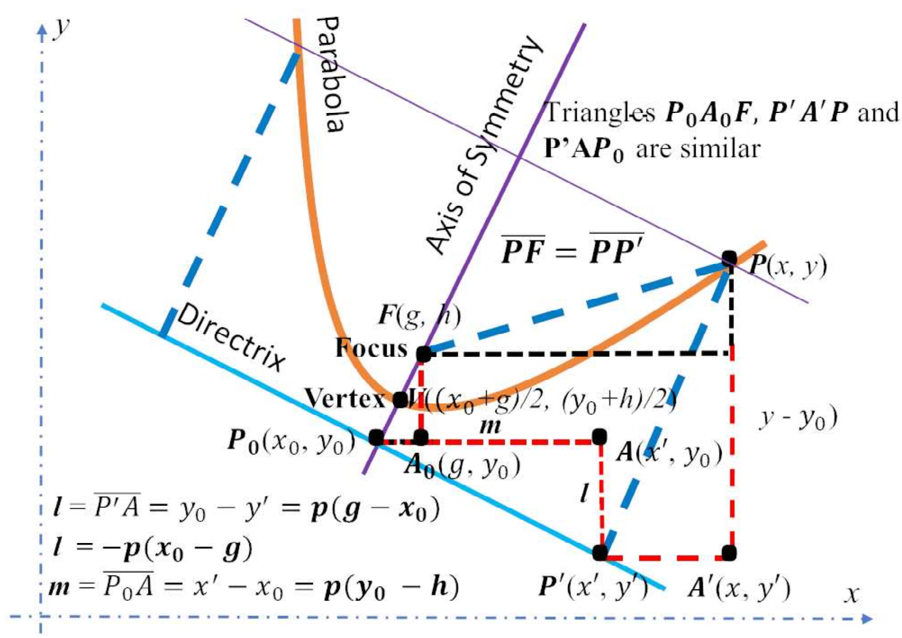


Figure 1: Relations Between the Parabola and its Directrix

$$(x-g)^2+\left(\frac{m}{l}(x-x_0)+y_0-h\right)^2=0 \qquad ...(2)$$

$$(lx-p_x)^2+(mx-q_x)^2=0\rightarrow x^2\left(l^2+m^2\right)-2\left(lp_x+mq_x\right)x+\left(p_x^2+q_x^2\right)=0 \qquad ...(3)$$

Solving and simplifying:

$$x=\frac{lp_x+mq_x\pm\sqrt{(lp_x+mq_x)^2-\left(l^2+m^2\right)\left(p_x^2+q_x^2\right)}}{l^2+m^2}=\frac{lp_z+mq_x\pm j\sqrt{(lq_x-mp_x)^2}}{l^2+m^2}$$

$$x=\frac{-\left(l^2+m^2\right)x_0-l\left[l(x_0-g)+m(y_0-h)\right]\pm jl\sqrt{\left(l^2+m^2\right)\left[(x_0-g)^2+(y_0-h)^2\right]-\left[l(x_0-g)+m(y_0-h)\right]^2}}{l^2+m^2} \qquad ...(4)$$

Notice that in Equation (4), the expression $l(x_0 - g) + m(y_0 - h)$ appears two times in the numerator. The way this expression was arrived at was: that which is within the square root was obtained by adding and subtracting the term, $l^2(x_0 - g)^2 + m^2(y_0 - h)^2$, see this below:

$$(lq_x-mp_x)^2=\left[l\left(mx_0-l(y_0-h)\right)-mlg\right]^2=l^2\left(m(x_0-g)-l(y_0-h)\right)^2$$

$$=l^2\left\{\left(m^2(x_0-g)^2+l^2(y_0-h)^2+\left[l^2(x_0-g)^2+m^2(y_0-h)^2\right]\right.\right.$$

$$\left.\left.-\left[l^2(x_0-g)^2+m^2(y_0-h)^2\right]-2m(x_0-g)l(y_0-h)\right)\right\}$$

$$(lq_x-mp_x)^2=l^2\left\{\left(l^2+m^2\right)\left[(x_0-g)^2+(y_0-h)^2\right]-\left[l(x_0-g)+m(y_0-h)\right]^2\right\} \qquad ...(5)$$

Furthermore, by adding and subtracting to the first member of the numerator, $(lp_x + mq_x)$, the term $l^2(x_0 - g)$, we obtain again the expression $l(x_0 - g) + m(y_0 - h)$ in it:

$$(lp_x+mq_x)=l^2g+m\left(mx_0-l(y_0-h)\right)=l^2g+m^2x_0-lm(y_0-h)=l^2g+m^2x_0+l^2(x_0-g)-lm(y_0-h)-l^2(x_0-g)$$

$$=\left(l^2+m^2\right)x_0-lm(y_0-h)-l^2(x_0-g)\ \ (lp_x+mq_x)=\left(l^2+m^2\right)x_0-l\left[l(x_0-g)+m(y_0-h)\right] \qquad ...(6)$$

And the Equation (4) is so obtained. Dividing numerator and denominator by $l^2 + m^2$: we get

$$x=x_0-\frac{l\left[l(x_0-g)+m(y_0-h)\right]}{\left(l^2+m^2\right)}\pm jl\sqrt{\frac{\left[(x_0-g)^2+(y_0-h)^2\right]}{\left(l^2+m^2\right)}-\frac{\left[l(x_0-g)+m(y_0-h)\right]^2}{\left(l^2+m^2\right)^2}}$$

The expression, $l(x_0 - g) + m(y_0 - h)$, which appears in the first term and in the radical sign of the numerator, is the dot product of the following two vectors: $l\mathbf{i} + m\mathbf{j}$ and $(x_0 - g)\mathbf{i} + (y_0 - h)\mathbf{j}$. Since they are the expressions of the directrix and parabola's axis they are perpendicular to each other, so their dot product must be zero. Thus, defining $a_0=\sqrt{(x_0-g)^2+(y_0-h)^2}$ and $b_0=\sqrt{l^2+m^2}$, and simplifying, the "cut" of the curve with its directrix is given by:

$$x=x_0\pm jl(a_0/b_0),\ y=y_0\pm jm(a_0/b_0) \qquad ...(7)$$

For example, in Figure 1, triangles $P'AP_0$ and P_0A_0P are proportional. By taking the values of l and m as $l = -p(y_0 - h)$ and $m = p(x_0 - g)$ where proportionality is ensured by the effect of the constant p, their dot product become null, as follows:

$$\left[-p(y_0-h)\mathbf{i}+p(x_0-g)\mathbf{j}\right]\bullet\left[(x_0-g)\mathbf{i}+(y_0-h)\mathbf{j}\right]=-p(y_0-h)(x_0-g)+p(x_0-g)(y_0-h)=0 \qquad ...(8)$$

With the last definitions we can write:

$$l^2 + m^2 = p^2(x_0 - g)^2 + p^2(y_0 - h)^2 = p^2 a_0^2 \qquad ...(9)$$

and the "cut off" in this case can be represented by:

$$\begin{Bmatrix} x = x_0 \pm jl/p, & y = y_0 \pm jm/p \\ x = x_0 \pm j[-(x_0 - g)], & y = y_0 \pm j(y_0 - h) \end{Bmatrix} \qquad ...(10)$$

Since imaginary parts behave as perpendicular vectors to the real ones, let's try next to identify them, in fact, as true vectors.

2.2 Interpretation of the Perpendicularity of Imaginary and Real Parts as Authentic Vector Axes

The perpendicularity between the imaginary and real parts of a complex number is an intuitive characteristic that works fine and is accepted worldwide. However, when this accepted feature, is diagramed in a drawing, the particle, $j = \sqrt{-1}$, does not appear affecting the representation of the imaginary and real parts of complex numbers. By considering these aspects in the definition of complex vectors, a mathematical interpretation could be: that the imaginary unit vector in the direction of the imaginary axis comes from the product of such particle, j, by the unit vector along the coordinate involved, i.e.: multiplying the particle, j, on the x axis by the unit vector **i**, and the same particle j, on the y axis by the unit vector **j**, would define the products, $j\mathbf{i} = \boldsymbol{i}_\perp$ and $j\mathbf{j} = \boldsymbol{j}_\perp$, as imaginary unit vectors perpendicular to **i**, and **j**, respectively, and also between them. Then, by doing these actions on the expressions of x and y in Equation (7), and after multiplying and reordering, we obtain as result a four-dimensional complex vector **r** composed by a real vector $\mathbf{r_0}$ and an imaginary one $\widehat{\boldsymbol{a}_0}$, in the following way:

$$\mathbf{r} = (x_0 \pm jl(a_0/b_0))\mathbf{i} + (y_0 \pm jm(a_0/b_0))\mathbf{j} = (x_0\mathbf{i} + y_0\mathbf{j}) \pm [l(j\mathbf{i}) + m(j\mathbf{j})](a_0/b_0) \qquad ...(11)$$

$$\mathbf{r} = (x_0\boldsymbol{i} + y_0\boldsymbol{j}) \pm [l(\boldsymbol{i}_\perp) + m(\boldsymbol{j}_\perp)](a_0/b_0) = \mathbf{r_0} \pm \widehat{\boldsymbol{a}_0} \qquad ...(12)$$

By remembering that: $l^2 + m^2 = b_0^2$ ...(13)

$$\left|\widehat{\boldsymbol{a}_0}\right| = \left|[l(j\mathbf{i}) \pm m(j\mathbf{j})](a_0/b_0)\right| = \left|[l(\boldsymbol{i}_\perp) \pm m(\boldsymbol{j}_\perp)](a_0/b_0)\right| = \sqrt{\frac{(l^2 + m^2)(a_0)^2}{(b_0)^2}} = a_0 \qquad ...(14)$$

$$\begin{Bmatrix} |\mathbf{r_0}| = \sqrt{x_0^2 + y_0^2}, \left|\widehat{\boldsymbol{a}_0}\right| = a_0 = \sqrt{(x_0 - g)^2 + (y_0 - h)^2} \\ |\mathbf{r}| = \sqrt{r_0^2 + a_0^2} = \sqrt{x_0^2 + y_0^2 + (x_0 - g)^2 + (y_0 - h)^2} \end{Bmatrix} \qquad ...(15)$$

Since the real unit vectors about the real axes, **i, j** and the imaginary unit vectors, about the imaginary axes $\boldsymbol{i}_\perp$, $\boldsymbol{j}_\perp$, define planes perpendicular to each other, with the origin O as the only common point, both resultant vectors, $\mathbf{r_0}$ and $\widehat{\boldsymbol{a}_0}$ are also perpendicular. Setting $\widehat{\boldsymbol{a}_0}$ as:

$$\widehat{\boldsymbol{a}_0} = [l(\boldsymbol{i}_\perp) \pm m(\boldsymbol{j}_\perp)](a_0/b_0) = [l(j\mathbf{i}) \pm m(j\mathbf{j})](a_0/b_0) = j(l\mathbf{i} \pm m\mathbf{j})(a_0/b_0) \qquad ...(16)$$

Namely, multiplication by j indicates that the resulting vector:

$$\widehat{\boldsymbol{a}_0} = j(l\mathbf{i} \pm m\mathbf{j})(a_0/b_0) \qquad ...(17)$$

is perpendicular to vector $l\mathbf{i} \pm m\mathbf{j}$, which is the direction of the directrix. So, $\widehat{\boldsymbol{a}_0}$ is perpendicular to the plane formed by directions $l\mathbf{i} \pm m\mathbf{j}$ and $\mathbf{r_0}$.

2.3 Cut of a Parabola with its Directrix in a Multi-Dimensional Space

Similarly, extending the cut of a parabola with its directrix line inside an N-dimensional space produces 2N-dimensional complex roots that can be expressed, with similar definitions to those of 2.1 and 2.2, as the following complex vector:

$$\mathbf{r} = \left(x_0\mathbf{i} \pm y_0\mathbf{j} \pm \cdots \pm z_0\mathbf{k}\right) + \left[l\left(\boldsymbol{i}_\perp\right) \pm m\left(\boldsymbol{j}_\perp\right) \pm \cdots \pm n\left(\boldsymbol{k}_\perp\right)\right]/\,p \to \mathbf{r} = \mathbf{r_0} + \widehat{\boldsymbol{a}_0} \quad ...(18)$$

Similarly, defining:

$$l^2 + m^2 + \cdots + n^2 = b_0^2; \left[\left(x_0 - g\right)^2 + \left(y_0 - h\right)^2 + \cdots + \left(z_0 - q\right)^2\right] = a_0^2 \quad ...(19)$$

$$\left|\mathbf{r_0}\right| = \sqrt{x_0^2 + y_0^2 + \cdots + z_0^2}; \left|\widehat{\boldsymbol{a}_0}\right| = \sqrt{\left(l^2 + m^2 + \cdots + n^2\right)\left(\frac{a_0}{b_0}\right)^2} = a_0;\ j\mathbf{i} = \boldsymbol{i}_\perp;\ j\mathbf{j} = \boldsymbol{j}_\perp; \cdots;\ j\mathbf{k} = \boldsymbol{k}_\perp \quad ...(20)$$

we have:

$$\left|\boldsymbol{r}\right| = \sqrt{r_0^2 + a_0^2} = \sqrt{x_0^2 + y_0^2 + \cdots + z_0^2 + \left(x_0 - g\right)^2 + \left(y_0 - h\right)^2 + \cdots + \left(z_0 - q\right)^2} \quad ...(21)$$

Since all unit vectors are perpendicular to each other, so are $\mathbf{r_0}$ and $\widehat{\boldsymbol{a}_0}$:

$$\widehat{\boldsymbol{a}_0} = \left[l\left(\boldsymbol{i}_\perp\right) \pm m\left(\boldsymbol{j}_\perp\right) \pm \cdots \pm n\left(\boldsymbol{k}_\perp\right)\right]\left(\frac{a_0}{b_0}\right) = j\left[l\mathbf{i} \pm m\mathbf{j} \pm \cdots \pm n\mathbf{k}\right]\left(\frac{a_0}{b_0}\right) \quad ...(22)$$

Thus, when direction (*l, m, ..., n*), is multiplied by the particle $j = \sqrt{-1}$, it becomes perpendicular to both the directrix and the axis of the parabola. General examples of complex vectors are developed next.

3. General Complex Vector Definition

Thus, from the results above an "N-dimensional" complex vector needs 2N dimensions to be algebraically represented and can be simply defined as:

$$\mathbf{r} = \left(a + jb\right)\mathbf{i} + \left(c + jd\right)\mathbf{j} + \cdots + \left(s + jt\right)\mathbf{n} = \left(a\mathbf{i} + c\mathbf{j} + \cdots + s\mathbf{n}\right) + \left(b\hat{\boldsymbol{i}}_\perp + d\hat{\boldsymbol{j}}_\perp + \cdots + t\hat{\boldsymbol{n}}_\perp\right) \quad ...(23)$$

$$\mathbf{r} = \boldsymbol{\rho} + \hat{\gamma},\ \boldsymbol{\rho} = a\mathbf{i} + c\mathbf{j} + \cdots + s\mathbf{n} = \rho\mathbf{u},\ \hat{\gamma} = b\hat{\boldsymbol{i}}_\perp + d\hat{\boldsymbol{j}}_\perp + \cdots + t\hat{\boldsymbol{n}}_\perp = \gamma\hat{\boldsymbol{u}}_\perp;\ \mathbf{r} = \rho\mathbf{u} + \hat{\gamma}\hat{\boldsymbol{u}}_\perp \quad ...(24)$$

$$\left\{\begin{matrix} \left|\boldsymbol{\rho}\right| = \rho = \sqrt{a^2 + c^2 + \cdots + s^2}; \left|\hat{\gamma}\right| = \hat{\gamma} = \sqrt{b^2 + d^2 + \cdots + t^2} \\ r = \sqrt{\rho^2 + \hat{\gamma}^2} = \sqrt{\left(a^2 + c^2 + \cdots + s^2\right) + \left(b^2 + d^2 + \cdots + t^2\right)} \end{matrix}\right\} \quad ...(25)$$

where, the perpendicular unit basis vectors: $\mathbf{i}$, $\hat{\boldsymbol{i}}_\perp$, $\mathbf{j}$, $\hat{\boldsymbol{j}}_\perp$, ..., $\mathbf{n}$, $\hat{\boldsymbol{n}}_\perp$, constitute a 2N-dimensional space: N real dimensions and N perpendicular imaginary ones, with a common origin O(0, 0, .., 0; 0, 0, ..,0). Likewise, given the perpendicularity between the real and imaginary N-dimensional unit vectors, the total resultant vectors, real $\boldsymbol{\rho} = \rho\mathbf{u}$, and imaginary $\hat{\gamma} = \gamma\hat{\boldsymbol{u}}_\perp$ are similarly perpendicular to each other. This defines $\mathbf{r} = \boldsymbol{\rho} + \hat{\gamma}$ as the total complex resultant vector, with its real and imaginary parts perpendicular to one another. Recalling some of the properties of the dot and cross product of pure vectors that can be applied and extended to complex vectors. From now on, the imaginary vector $\hat{\boldsymbol{i}}_\perp$ will be written without the subscript that indicated perpendicularity, leaving only the italic format to simplify its writing, $\hat{\boldsymbol{i}}_\perp \to \hat{\boldsymbol{i}}$.

4. Characteristics of Dot Product and Cross Product of Two Vectors

4.1. Multi-Dimensional Scalar or Dot Product of Two Pure Vectors

As it is known, the scalar product of two vectors **a** and **b** is a scalar number defined as the product of their magnitudes multiplied by the cosine of the angle ϕ between the two vectors:

$$\mathbf{a} \cdot \mathbf{b} = \left|\mathbf{a}\right|\left|\mathbf{b}\right|\cos\phi \quad ...(26)$$

And on the other hand, in an N-dimensional space, for $\mathbf{a} = (a_x, a_y, ..., a_z)$ and $\mathbf{b} = (b_x, b_y, ..., b_z)$:

$$\mathbf{a} \cdot \mathbf{b} = a_x b_x + a_y b_y + \cdots + a_z b_z \quad ...(27)$$

Where, $\left|\mathbf{a}\right| = \sqrt{a_x^2 + a_y^2 + \cdots + a_z^2}$ and $\left|\mathbf{b}\right| = \sqrt{b_x^2 + b_y^2 + \cdots + b_z^2}$ ...(28)

4.2 Multi-Dimensional Cross or Vector Product of Two Pure Vectors

The cross product of two vectors **a** and **b**, forming an angle ϕ between them, can be defined, as a vector *c* perpendicular to **a** and **b** with the following features:

$$\mathbf{c} = \mathbf{a}\times\mathbf{b} = |\mathbf{a}||\mathbf{b}|\sin\phi\mathbf{n} = |\mathbf{a}||\mathbf{b}|\sqrt{1-\cos^2\phi}\mathbf{n} = \sqrt{|\mathbf{a}|^2|\mathbf{b}|^2-(\mathbf{a}\cdot\mathbf{b})^2}\mathbf{n} \quad ...(29)$$

for $|\mathbf{c}| = |\mathbf{a}||\mathbf{b}|\sin\phi = |\mathbf{a}||\mathbf{b}|\sqrt{1-\cos^2\phi} = \sqrt{|\mathbf{a}|^2|\mathbf{b}|^2-(\mathbf{a}\cdot\mathbf{b})^2}$ ...(30)

Thus, any different approach of the multidimensional product of two pure vectors must have a result equal to Equations (29) and (30) as is demonstrated next in an N-dimensional space with the following expression:

$$\left\{\begin{array}{c} unit\ vectors = \mathbf{i},\ \mathbf{j},\ \mathbf{v},\ \ldots,\ \mathbf{w},\ \mathbf{k} \\ \mathbf{a} = (a_x,\ a_y,\ a_v,\ \ldots,\ a_w,\ a_z) \\ \mathbf{b} = (b_x,\ b_y,\ b_v,\ \ldots,\ b_w,\ b_z) \end{array}\right\}\left\{\begin{array}{l} |\mathbf{a}| = \sqrt{a_x^2+a_y^2+a_v^2+\cdots+a_w^2+a_z^2} \\ |\mathbf{b}| = \sqrt{b_x^2+b_y^2+b_v^2+\cdots+b_w^2+b_z^2} \end{array}\right\} \quad ...(31)$$

The cross product can also be defined as the sum of $C_{N,2} = N(N-1)/2$square matrices, 2x2 each multiplied by a multivector $\mathbf{u}_{]pq[}$ perpendicular to the unit vectors **p**, **q**, containing the remaining N – 2 unit vectors. Multivector $\mathbf{u}_{]pq[}$ is constructed by eliminating N – 2 columns and the first row, and after placing the first column in the last position and repeating the same structural calculation:

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{v} & \cdots & \mathbf{w} & \mathbf{k} \\ a_x & a_y & a_v & \cdots & a_w & a_z \\ b_x & b_y & b_v & \cdots & b_w & b_z \end{vmatrix} \quad ...(32)$$

For:

$\mathbf{u}_{]ij[} = \mathbf{v}, ..., \mathbf{w}, \mathbf{k}$; $\mathbf{u}_{]iv[} = \mathbf{j}, ..., \mathbf{w}, \mathbf{k}$; ...; $\mathbf{u}_{]iw[} = \mathbf{j}, \mathbf{v}, ..., \mathbf{k}$; $\mathbf{u}_{]ik[} = \mathbf{j}, \mathbf{v}, ..., \mathbf{w}$; $\mathbf{u}_{]jv[} = \mathbf{i}, ..., \mathbf{w}, \mathbf{k}$; ... ...; $\mathbf{u}_{\mathbf{wk}} = \mathbf{i}, \mathbf{j}, \mathbf{v}, ...$;

where the next equality must be satisfied (as indeed it is):

$$\mathbf{c} = \mathbf{a}\times\mathbf{b} = \sqrt{|\mathbf{a}|^2|\mathbf{b}|^2-(\mathbf{a}\cdot\mathbf{b})^2}\mathbf{n} = \begin{vmatrix} a_x & a_y \\ b_x & b_y \end{vmatrix}\mathbf{u}_{]ij[} + \begin{vmatrix} a_x & a_v \\ b_x & b_v \end{vmatrix}\mathbf{u}_{]iv[} + \cdots + \begin{vmatrix} a_x & a_z \\ b_x & b_z \end{vmatrix}\mathbf{u}_{]ik[} + \begin{vmatrix} a_y & a_w \\ b_y & b_w \end{vmatrix}\mathbf{u}_{]jw[} + \cdots + \begin{vmatrix} a_w & a_z \\ b_w & b_z \end{vmatrix}\mathbf{u}_{]wk[} \quad ...(33)$$

$$\mathbf{c} = (a_xb_y - b_xa_y)\mathbf{u}_{]ij[} + (a_xb_v - b_xa_y)\mathbf{u}_{]iv[} + (a_xb_z - b_xa_z)\mathbf{u}_{]ik[} + \cdots + (a_wb_z - b_wa_z)\mathbf{u}_{]wk[} \quad ...(34)$$

Absolute value of **c** (that is $\lfloor\mathbf{c}\rfloor$) also can be calculated from above as:

$$\mathbf{c} = \mathbf{a}\times\mathbf{b} = |\mathbf{c}|\mathbf{n} = \sqrt{(a_xb_y-b_xa_y)^2+(a_xb_v-b_xa_v)^2+(a_xb_z-b_xa_z)^2+\cdots+(a_wb_z-b_2a_z)^2}\mathbf{n} \quad ...(35)$$

This was checked with Sagemath software verifying that the following is null:

$$|\mathbf{a}|^2|\mathbf{b}|^2-(\mathbf{a}\cdot\mathbf{b})^2-\left[(a_xb_y-b_xa_y)^2+(a_xb_v-b_xa_v)^2+(a_xb_z-b_xa_z)^2+\cdots+(a_wb_z-b_wa_z)^2\right]=0 \quad ...(36)$$

5. General Dot and Cross Product of Two Complex Vectors

Previous definitions obtained for pure vectors can be applied directly to complex vectors. Let's see some examples, to realize about other features of complex vectors' dot and cross product.

5.1 Dot and Cross Product of TWO Complex Vectors in a Space of Two Dimensions

For example, let two complex vectors be: $\mathbf{r_1} = (x_{11}+jy_{11})\mathbf{i} + (x_{12}+jy_{12})\mathbf{j}$ and $\mathbf{r_2} = (x_{21}+jy_{21})\mathbf{i} + (x_{22}+jy_{22})\mathbf{j}$. Its dot product, converted and reordered, becomes:

$$\boldsymbol{r_1}\cdot\boldsymbol{r_2} = \left[(x_{11}+jy_{11})\boldsymbol{i}+(x_{12}+jy_{12})\boldsymbol{j}\right]\cdot\left[(x_{21}+jy_{21})\boldsymbol{i}+(x_{22}+jy_{22})\boldsymbol{j}\right] = \left[x_{11}\boldsymbol{i}+x_{12}\boldsymbol{j}+y_{11}\hat{\boldsymbol{i}}+y_{12}\hat{\boldsymbol{j}}\right]\cdot\left[x_{21}\boldsymbol{i}+x_{22}j+y_{21}\hat{i}+y_{22}\hat{j}\right]$$

$$= x_{11}x_{21}+x_{12}x_{22}+y_{11}y_{21}+y_{12}y_{22} \quad ...(37)$$

The cross or vector product of these two complex vectors, reordered into a 4D-space, arrives at:

$$\mathbf{r_1}\times\mathbf{r_2}=\left[(x_{11}+jy_{11})\mathbf{i}+(x_{12}+jy_{12})\mathbf{j}\right]\times\left[(x_{21}+jy_{21})\mathbf{i}+(x_{22}+jy_{22})\mathbf{j}\right]=\left[x_{11}\mathbf{i}+x_{12}\mathbf{j}+y_{11}\hat{\boldsymbol{i}}+y_{12}\hat{\boldsymbol{j}}\right]\times\left[x_{21}\mathbf{i}+x_{22}\mathbf{j}+y_{21}\hat{\boldsymbol{i}}+y_{22}\hat{\boldsymbol{j}}\right] \quad ...(38)$$

$$\mathbf{r_1}\times\mathbf{r_2}=(x_{11}x_{22}-x_{12}x_{21})(\mathbf{i}\times\mathbf{j})+(x_{11}y_{21}-y_{11}x_{21})(\mathbf{i}\times\hat{\boldsymbol{i}})+(x_{11}y_{22}-y_{12}x_{21})(\mathbf{i}\times\hat{\boldsymbol{j}})+(x_{12}y_{21}-y_{11}x_{22})(\mathbf{j}\times\hat{\boldsymbol{i}})+(x_{12}y_{22}-y_{12}x_{22})(\mathbf{j}\times\hat{\boldsymbol{j}})$$
$$+(y_{11}y_{22}-y_{12}y_{21})(\hat{\boldsymbol{i}}\times\hat{\boldsymbol{j}}) \quad ...(39)$$

where, **i**, **j**, are the real unit vectors, and, $j\mathbf{i}=\hat{\imath}$, $j\mathbf{j}=\hat{\jmath}$, are the imaginary ones of the 4D axes, all perpendicular among them. Expressing the following $C_{4,2}$ = 6 cross products based on the six unit vectors: **i** x **j** = $\mathbf{n_{ij}}=\mathbf{u_{]ij[}}=(\hat{\imath},\hat{\jmath})$, **i** x $\hat{\imath}$ = $\mathbf{n}_{\mathbf{i}\hat{\imath}}=\mathbf{u}_{]\mathbf{i}\hat{\imath}[}=(\mathbf{j},\hat{\jmath})$, **i** x $\hat{\jmath}$ = $\mathbf{n}_{\mathbf{i}\hat{\jmath}}=\mathbf{u}_{]\mathbf{i}\hat{\jmath}[}=(\hat{\imath},\mathbf{j})$, **j** x $\hat{\imath}$ = $\mathbf{n}_{\mathbf{j}\hat{\imath}}=\mathbf{u}_{]\mathbf{j}\hat{\imath}[}=(\mathbf{i},\hat{\boldsymbol{j}})$, **j** x $\hat{\jmath}$ = $\mathbf{n}_{\mathbf{j}\hat{\jmath}}=\mathbf{u}_{]\mathbf{j}\hat{\jmath}[}=(\mathbf{i},\hat{\imath})$, $\hat{\imath}$ x $\hat{\jmath}$ = $\mathbf{n}_{\hat{\imath}\hat{\jmath}}=\mathbf{u}_{]\hat{\imath}\hat{\jmath}[}=(\mathbf{i},\mathbf{j})$, for]**p**,**q**[≠ (**p**,**q**). Thus, the complex vector **c** = $\mathbf{r_1}$ x $\mathbf{r_2}$, reorders to the following expression:

$$\mathbf{c}=\mathbf{r_1}\times\mathbf{r_2}=(x_{11}x_{22}-x_{12}x_{21})\mathbf{n_{ij}}+(x_{11}y_{21}-y_{11}x_{21})\mathbf{n}_{\mathbf{i}\hat{\imath}}+(x_{11}y_{22}-y_{12}x_{21})\mathbf{n}_{\mathbf{i}\hat{\jmath}}+(x_{12}y_{21}-y_{11}x_{22})\mathbf{n}_{\mathbf{j}\hat{\imath}}+(x_{12}y_{22}-y_{12}x_{22})\mathbf{n}_{\mathbf{j}\hat{\jmath}}+(y_{11}y_{22}-y_{12}y_{21})\mathbf{n}_{\hat{\imath}\hat{\jmath}} \quad ...(40)$$

$$\mathbf{c}=\mathbf{r_1}\times\mathbf{r_2}=\sqrt{(x_{11}x_{22}-x_{12}x_{21})^2+(x_{11}y_{21}-x_{21}y_{11})^2+(x_{11}y_{22}-y_{12}x_{21})^2+(x_{12}y_{21}-y_{11}x_{22})^2+(x_{12}y_{22}-y_{12}x_{22})^2+(y_{11}y_{22}-y_{12}y_{21})}\mathbf{n} \quad ...(41)$$

where **c** is perpendicular to $\mathbf{r_1}$ and $\mathbf{r_2}$ and **n** is a complex unit vector with equal direction as **c**. Notice that products of unit basis **i**, **j**, $\hat{\imath}$, $\hat{\jmath}$, creates a Six-Dimensional space with 6 unit-basis vectors $\mathbf{n_{p,q}}$ = **p** x **q** = $\mathbf{u_{]p,q[}}$, for]**p**,**q**[= the remaining two unit vectors, different of **p**, **q**; perpendicular among them and to **p, q**. An easy way to construct and calculate the terms in parentheses in Equation (41) is to use the 3x4 matrix arrangement:

$$\begin{vmatrix}\mathbf{i} & \mathbf{j} & \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}}\\ x_{11} & x_{12} & y_{11} & y_{12}\\ x_{21} & x_{22} & y_{21} & y_{22}\end{vmatrix} \quad ...(42)$$

The "2D" cross product reordered to as a 4D product becomes equal to the number of $C_{4,2}$ = 6 combinations, put alike into matrix form, where the first row indicates the referred unit vector and the involved columns (**p**, **q**) (notice that the six $\mathbf{n_{pq}}=\mathbf{u_{]pq[}}$ = (**r**, **t**), for **r**, **t** ≠ **p**, **q** bases generate a six-dimensional space), with the sign of the 2x2 matrices in ascending order, namely, by eliminating the first row and the (**r**, **t**) columns, we have:

$$\mathbf{c}=\mathbf{r_1}\times\mathbf{r_2}=\begin{vmatrix}x_{11} & x_{12}\\ x_{21} & x_{22}\end{vmatrix}\mathbf{u}_{\hat{\imath}\hat{\jmath}}-\begin{vmatrix}x_{11} & y_{11}\\ x_{21} & x_{21}\end{vmatrix}\mathbf{u}_{\mathbf{i}\hat{\jmath}}+\begin{vmatrix}x_{11} & y_{12}\\ x_{21} & y_{22}\end{vmatrix}\mathbf{u}_{\mathbf{j}\hat{\imath}}+\begin{vmatrix}x_{12} & y_{11}\\ x_{22} & y_{21}\end{vmatrix}\mathbf{u}_{\mathbf{i}\hat{\jmath}}-\begin{vmatrix}x_{12} & y_{12}\\ x_{22} & y_{22}\end{vmatrix}\mathbf{u}_{\mathbf{i}\hat{\imath}}+\begin{vmatrix}y_{11} & y_{12}\\ y_{21} & y_{22}\end{vmatrix}\mathbf{u_{ij}} \quad ...(43)$$

To prove Equations (41), (42) and (43), according to the definition of the cross product, $\mathbf{r_1}$ x $\mathbf{r_2}$ = $r_1r_2[\sin(\theta_2-\theta_1)]\mathbf{n}$ = $r_1r_2\sin\theta\,\mathbf{n}$, for $\mathbf{r_1}=x_{11}\mathbf{i}+x_{12}\mathbf{j}+y_{11}\hat{\imath}+y_{12}\hat{\jmath}$ and $\mathbf{r_2}=x_{21}\mathbf{i}+x_{22}\mathbf{j}+y_{21}\hat{\imath}+y_{22}\hat{\jmath}$, and reordering, we can check the double equality:

$$\mathbf{c}=\mathbf{r_1}\times\mathbf{r_2}=\sqrt{(r_1r_2)^2-(r_1\cdot r_2)^2}\,\mathbf{n}=\sqrt{\begin{matrix}(x_{11}^2+x_{12}^2+y_{11}^2+y_{12}^2)*(x_{21}^2+x_{22}^2+y_{21}^2+y_{22}^2)\\ -(x_{11}x_{21}+x_{12}x_{22}+y_{11}y_{21}+y_{12}y_{22})^2\end{matrix}}\,\mathbf{n} \quad ...(44)$$

Or, establishing that the expression inside the square root in Equaiton (44) minus that of the square root in Equation (41), by using the help of Sagemath software, is easy to show that such difference is null:

var('x_11,x_12,x_21,x_22, y_11, y_12,y_21,y_22')

expand((((x_11)^2+(x_12)^2+(y_11)^2+(y_12)^2)*((x_21)^2+(x_22)^2+(y_21)^2+(y_22)^2) - ((x_11*x_21 + x_12*x_22 + y_11*y_21 + y_12*y_22)^2)) ((x_11*x_22 - x_12*x_21)^2 + (x_11*y_21 - x_21*y_11)^2 + (x_11*y_22 - y_12*x_21)^2 + (x_12*y_21 - y_11*x_22)^2 + (x_12*y_22 - y_12*x_22)^2 + (y_11*y_22 - y_12*y_21)^2)) == 0 ...(45)

It is worth noticing that vector **c**, can be reduced to two terms: $a\mathbf{u}_\mathbf{R}$, a real vector constituted by the terms multiplying the real-basis vectors $\mathbf{n}_{ij} = \mathbf{i} \times \mathbf{j}$ and $\mathbf{n}_{\hat{i}\hat{j}} = \hat{\boldsymbol{i}} \times \hat{\boldsymbol{j}}$. The last unit basis should become a real one perpendicular to **i**, **j** and of course to $\hat{\boldsymbol{i}}$, $\hat{\boldsymbol{j}}$, because it does not contain internally the factor *j*, but the factor $j^2 = -1$, which is a real number and it should not create an imaginary new axis, but a real new axis. Thus, it suggests us that we can define it as a new real unit basis as corresponding to a component of $a\mathbf{u}_{\mathbf{real}}$ without the factor *j*. The term $b\hat{\boldsymbol{u}}_{imag}$ is built by the remaining four imaginary unit bases containing the factor *j*. In this way,

$$\mathbf{c} = \mathbf{r_1} \times \mathbf{r_2} = a\mathbf{u_R} + b\hat{\mathbf{u}}_I = a\mathbf{u_R} + b(j\mathbf{u_I}) = a\mathbf{u_R} + jb\mathbf{u_I} \qquad ...(46)$$

Thus, we have returned the change $\mathbf{c} = a\mathbf{u_R} + b\hat{\mathbf{u}}_{\boldsymbol{I}}$, to $\mathbf{c} = a\mathbf{u_R} + jb\mathbf{u_I}$, in order to emphasize the perpendicularity of vectors $\mathbf{u_R}$ and $\hat{\boldsymbol{u}}_{\boldsymbol{I}} = \boldsymbol{u}_{\boldsymbol{I}} = j\mathbf{u_I}$. So, expressions of $\mathrm{a}\mathbf{u_R}$ and $\mathrm{b}\boldsymbol{u}_{\boldsymbol{I}}$ in (49), from (46), explicitly become:

$$a\mathbf{u_R} = (x_{11}x_{22} - x_{12}x_{21})\mathbf{u_{ij}} + (y_{11}y_{22} - y_{12}y_{21})\,\mathbf{u}_{\hat{i}\hat{j}} \qquad ...(47)$$

$$\mathrm{b}\boldsymbol{u}_{\boldsymbol{I}} = (x_{11}y_{21} - x_{12}y_{11})\mathbf{u}_{\mathbf{i}\hat{i}} + (x_{11}y_{22} - x_{21}\,y_{12})\mathbf{u}_{\mathbf{i}\hat{j}} + (x_{12}y_{21} - x_{22}y_{11})\mathbf{u}_{\mathbf{j}\hat{i}} + (x_{12}y_{22} - x_{22}y_{12})\mathbf{u}_{\mathbf{j}\hat{j}} \qquad ...(48)$$

$$a\mathbf{u_R} = \sqrt{\left(x_{11}x_{22} - x_{12}x_{21}\right)^2 + \left(y_{11}y_{22} - y_{21}y_{12}\right)^2}\,\mathbf{u_R} \qquad ...(49)$$

$$b\boldsymbol{u}_{\boldsymbol{I}} = \sqrt{\left(x_{11}y_{21} - x_{12}y_{11}\right)^2 + \left(x_{11}y_{22} - x_{12}y_{21}\right)^2 + \left(x_{12}y_{21} - x_{11}y_{22}\right)^2 + \left(x_{12}y_{22} - y_{12}x_{22}\right)^2}\,\boldsymbol{u}_{\boldsymbol{I}} \qquad ...(50)$$

Observe that in the case of pure vectors, without imaginary axes, the expression of $\boldsymbol{c}$ reduces to:

$$\mathbf{c} = \mathbf{r_1} \times \mathbf{r_2} = \begin{vmatrix} \mathbf{k} & \mathbf{i} & \mathbf{j} \\ 0 & x_{11} & x_{12} \\ 0 & x_{21} & x_{22} \end{vmatrix} = \begin{vmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{vmatrix}\mathbf{k} = \left(x_{11}x_{22} - x_{21}x_{12}\right)\mathbf{k}$$

So, it simplifies consistently to the cross product of two pure vectors in a 2-Dimensional space.

5.2 Dot and Cross Product of TWO Complex Vectors in a 3-Dimensional Space

Let's apply the same procedure to the cross product of the 3D-complex vectors, $\mathbf{r_1}$ and $\mathbf{r_2}$, for $\mathbf{r_1} = (x_{11} + jy_{11})\mathbf{i} + (x_{12} + jy_{12})\mathbf{j} + (x_{13} + jy_{13})\mathbf{k}$ and $\mathbf{r_2} = (x_{21} + jy_{21})\mathbf{i} + (x_{22} + jy_{22})\mathbf{j} + (x_{23} + jy_{23})\mathbf{k}$.

The dot product can be reordered and written as:

$$\mathbf{r_1} \cdot \mathbf{r_2} = \left[\left(x_{11} + jy_{11}\right)\mathbf{i} + \left(x_{12} + jy_{12}\right)\mathbf{j} + \left(x_{13} + jy_{13}\right)\mathbf{k}\right] \cdot \left[\left(x_{21} + jy_{21}\right)\mathbf{i} + \left(x_{22} + jy_{22}\right)\mathbf{j} + \left(x_{23} + jy_{23}\right)\mathbf{k}\right]$$

$$\mathbf{r_1} \cdot \mathbf{r_2} = \left[x_{11}\mathbf{i} + x_{12}\mathbf{j} + x_{13}\mathbf{k} + y_{11}\hat{\boldsymbol{i}} + y_{12}\hat{\boldsymbol{j}} + y_{13}\hat{\boldsymbol{k}}\right] \cdot \left[x_{21}\mathbf{i} + x_{22}\mathbf{j} + x_{23}\mathbf{k} + y_{21}\hat{\boldsymbol{\imath}} + y_{22}\hat{\boldsymbol{\jmath}} + y_{23}\hat{\boldsymbol{k}}\right]$$

$$\mathbf{r_1} \cdot \mathbf{r_2} = x_{11}x_{21} + x_{12}x_{22} + x_{13}x_{23} + y_{11}y_{21} + y_{12}y_{22} + y_{13}y_{23} \qquad ...(51)$$

For the cross or vector product of two complex vectors in three dimensions (actually six: three real, and three imaginary axes, as it was within the dot product) we have:

$$\mathbf{r_1} \times \mathbf{r_2} = \left[\left(x_{11} + jy_{11}\right)\mathbf{i} + \left(x_{12} + jy_{12}\right)\mathbf{j} + \left(x_{13} + jy_{13}\right)\mathbf{k}\right] \times \left[\left(x_{21} + jy_{21}\right)\mathbf{i} + \left(x_{22} + jy_{22}\right)\mathbf{j} + \left(x_{23} + jy_{23}\right)\mathbf{k}\right]$$

$$= \left[x_{11}\mathbf{i} + x_{12}\mathbf{j} + x_{13}\mathbf{k} + y_{11}\hat{\boldsymbol{i}} + y_{12}\hat{\boldsymbol{j}} + y_{13}\hat{\boldsymbol{k}}\right] \times \left[x_{21}\mathbf{i} + x_{22}\mathbf{j} + x_{23}\mathbf{k} + y_{21}\hat{\boldsymbol{i}} + y_{22}\hat{\boldsymbol{j}} + y_{23}\hat{\boldsymbol{k}}\right]$$

With a similar cross product layout guide, by eliminating N–2 columns and the first row, as in Equation (42): $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} & \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ x_{11} & x_{12} & x_{13} & y_{11} & y_{12} & y_{13} \\ x_{21} & x_{22} & x_{23} & y_{21} & y_{22} & y_{23} \end{vmatrix}$, its order is as follows:

$$\mathbf{c} = \mathbf{r_1} \times \mathbf{r_2} = \begin{vmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{vmatrix}\mathbf{u}_{]\mathbf{ij}[} - \begin{vmatrix} x_{11} & x_{13} \\ x_{21} & x_{23} \end{vmatrix}\mathbf{u}_{]\mathbf{ik}[} + \begin{vmatrix} x_{11} & y_{11} \\ x_{21} & y_{21} \end{vmatrix}\mathbf{u}_{]\mathbf{i}\hat{i}[} - \begin{vmatrix} x_{11} & y_{12} \\ x_{21} & y_{22} \end{vmatrix}\mathbf{u}_{]\mathbf{i}\hat{j}[} + \begin{vmatrix} x_{11} & y_{11} \\ x_{21} & y_{21} \end{vmatrix}\mathbf{u}_{]\mathbf{i}\hat{k}[} - \begin{vmatrix} x_{12} & x_{13} \\ x_{22} & x_{23} \end{vmatrix}\mathbf{u}_{]\mathbf{jk}[} - \begin{vmatrix} x_{12} & y_{11} \\ x_{22} & y_{21} \end{vmatrix}\mathbf{u}_{]\mathbf{j}\hat{i}[} + \begin{vmatrix} x_{12} & y_{12} \\ x_{22} & y_{22} \end{vmatrix}\mathbf{u}_{]\mathbf{j}\hat{j}[}$$

$$+ \begin{vmatrix} x_{12} & y_{13} \\ x_{22} & y_{23} \end{vmatrix}\mathbf{u}_{]\mathbf{j}\hat{k}[} + \begin{vmatrix} x_{13} & y_{11} \\ x_{23} & y_{21} \end{vmatrix}\mathbf{u}_{]\mathbf{k}\hat{i}[} + \begin{vmatrix} x_{13} & y_{12} \\ x_{23} & y_{22} \end{vmatrix}\mathbf{u}_{]\mathbf{k}\hat{j}[} + \begin{vmatrix} x_{13} & y_{13} \\ x_{23} & y_{23} \end{vmatrix}\mathbf{u}_{]\mathbf{k}\hat{k}[} + \begin{vmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{vmatrix}\mathbf{u}_{]\hat{i}\hat{j}[} - \begin{vmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{vmatrix}\mathbf{u}_{]\hat{i}\hat{k}[} + \begin{vmatrix} y_{12} & y_{13} \\ y_{22} & y_{23} \end{vmatrix}\mathbf{u}_{]\hat{j}\hat{k}[} \qquad ...(52)$$

Where for example $\mathbf{u}_{]\mathbf{j}\hat{k}[}=\left(\mathbf{i},\,\mathbf{k},\,\hat{\imath},\,\hat{\jmath}\right)$. The number of 2x2 matrices given by the combination of 6 dimensions taken in pairs, is: $\boldsymbol{C}_{\mathbf{6,2}}=\mathbf{15}$, resulting finally a Fifteen-Dimensional space,

$$\begin{aligned}\mathbf{c}=\mathbf{r_1}\times\mathbf{r_2}=&\left(x_{11}x_{22}-x_{21}x_{12}\right)\mathbf{u}_{]\mathbf{ij}[}-\left(x_{11}x_{23}-x_{21}x_{13}\right)\mathbf{u}_{]\mathbf{ik}[}+\left(x_{11}y_{21}-x_{21}y_{11}\right)\mathbf{u}_{]\mathbf{i}\hat{\imath}[}-\left(x_{11}y_{22}-x_{21}y_{12}\right)\mathbf{u}_{]\mathbf{i}\hat{\jmath}[}+\left(x_{11}y_{23}-x_{21}y_{13}\right)\mathbf{u}_{]\mathbf{i}\hat{k}[}\\&+\left(x_{12}x_{23}-x_{22}x_{13}\right)\mathbf{u}_{]\mathbf{jk}[}-\left(x_{12}y_{21}-x_{22}y_{11}\right)\mathbf{u}_{]\mathbf{j}\hat{\imath}[}+\left(x_{12}y_{22}-x_{22}y_{12}\right)\mathbf{u}_{]\mathbf{j}\hat{\jmath}[}-\left(x_{12}y_{23}-x_{22}y_{13}\right)\mathbf{u}_{]\mathbf{j}\hat{k}[}+\left(x_{13}y_{21}-x_{23}y_{11}\right)\mathbf{u}_{]\mathbf{k}\hat{\imath}[}\\&-\left(x_{13}y_{22}-x_{23}y_{12}\right)\mathbf{u}_{]\mathbf{k}\hat{\jmath}[}+\left(x_{13}y_{23}-x_{23}y_{13}\right)\mathbf{u}_{]\mathbf{k}\hat{k}[}+\left(y_{11}y_{22}-y_{21}y_{12}\right)\mathbf{u}_{]\hat{\imath}\hat{\jmath}[}-\left(y_{11}y_{23}-y_{21}y_{23}\right)\mathbf{u}_{]\hat{\imath}\hat{k}[}+\left(y_{12}y_{23}-y_{22}y_{13}\right)\mathbf{u}_{]\hat{\jmath}\hat{k}[}\end{aligned}\qquad ...(53)$$

Direction of resultant vector **c** is simultaneously perpendicular to: vectors $\mathbf{r_1}$ and $\mathbf{r_2}$, all unit vectors $\mathbf{u}_{]\mathbf{mn}[}$, for]**m, n**[= **p**, **q**, **r**, **s**; all basis vectors $\mathbf{i},\,\mathbf{j},\,\mathbf{k},\,\hat{\imath},\,\hat{\jmath},\,\hat{k}$ for any value of **m**, **n**, **p**, **q**, **r**, **s** = $\mathbf{i},\,\mathbf{j},\,\mathbf{k},\,\hat{\imath},\,\hat{\jmath},\,\hat{k}$. For instance, $\mathbf{u}_{]\mathbf{jk}[}=\left(\mathbf{i},\,\hat{\imath},\,\hat{\jmath},\,\hat{k}\right)$. So:

$$\mathbf{c}=\mathbf{r_1}\times\mathbf{r_2}=\sqrt{\begin{aligned}&\left(x_{11}x_{22}-x_{21}x_{12}\right)^2+\left(x_{11}x_{23}-x_{21}x_{13}\right)^2+\left(x_{11}y_{21}-x_{21}y_{11}\right)^2+\left(x_{11}y_{22}-x_{21}y_{12}\right)^2+\left(x_{11}y_{23}-x_{21}y_{13}\right)^2\\&+\left(x_{12}x_{23}-x_{22}x_{13}\right)^2+\left(x_{12}y_{21}-x_{22}y_{11}\right)^2+\left(x_{12}y_{22}-x_{22}y_{12}\right)^2+\left(x_{12}y_{23}-x_{22}y_{13}\right)^2+\left(x_{13}y_{21}-x_{23}y_{11}\right)^2\\&+\left(x_{13}y_{22}-x_{23}y_{12}\right)^2+\left(x_{13}y_{23}-x_{23}y_{13}\right)^2+\left(y_{11}y_{22}-y_{21}y_{12}\right)^2+\left(y_{11}y_{23}-y_{21}y_{13}\right)^2+\left(y_{12}y_{23}-y_{22}y_{13}\right)^2\end{aligned}}\;\mathbf{n}\qquad ...(54)$$

Checking, as before, this output via: $\mathbf{c}=\mathbf{r_1}\times\mathbf{r_2}=r_1r_2\sin\theta\,\mathbf{n}=\sqrt{\left(r_1r_2\right)^2-\left(r_1\cdot r_2\right)^2}\,\mathbf{n}$, for $\mathbf{r_1}=(x_{11}+jy_{11})\mathbf{i}+(x_{12}+jy_{12})\mathbf{j}+(x_{13}+jy_{13})\mathbf{k}$] and $\mathbf{r_2}=[(x_{21}+jy_{21})\mathbf{i}+(x_{22}+jy_{22})\mathbf{j}+(x_{23}+jy_{23})\mathbf{k}$, we obtain:

$$\mathbf{c}=\mathbf{r_1}\times\mathbf{r_2}=\sqrt{\left(r_1r_2\right)^2-\left(r_1\cdot r_2\right)^2}\,\mathbf{n}$$

$$=\sqrt{\begin{aligned}&\left(x_{11}^2+x_{12}^2+x_{13}^2+y_{11}^2+y_{12}^2+y_{13}^2\right)\left(x_{21}^2+x_{22}^2+x_{23}^2+y_{21}^2+y_{22}^2+y_{23}^2\right)\\&-\left(x_{11}x_{21}+x_{12}x_{22}+x_{13}x_{23}+y_{11}y_{21}+y_{12}y_{22}+y_{13}y_{23}\right)^2\end{aligned}}\;\mathbf{n}\qquad ...(55)$$

expand((((x_11)^2+(x_12)^2+(x_13)^2+(y_11)^2+(y_12)^2+(y_13)^2)*((x_21)^2+(x_22)^2+(x_23)^2+(y_21)^2+(y_22)^2+(y_23)^2) - (x_11*x_21 + x_12*x_22 + x_13*x_23 + y_11*y_21 + y_12*y_22 + y_13*y_23)^2 - ((x_11*x_22 - x_21*x_12)^2 + (x_11*x_23 - x_21*x_13)^2 + (x_11*y_21 - x_21*y_11)^2 + (x_11*y_22 - x_21*y_12)^2 + (x_11*y_23 - x_21*y_13)^2 + (x_12*x_23 - x_22*x_13)^2 + (x_12*y_21 - x_22*y_11)^2 + (x_12*y_22 - x_22*y_12)^2 + (x_12*y_23 - x_22*y_13)^2 + (x_13*y_21 - x_23*y_11)^2 + (x_13*y_22 - x_23*y_12)^2 + (x_13*y_23 - x_23*y_13)^2 + (y_11*y_22 - y_21*y_12)^2 + (y_11*y_23 - y_21*y_13)^2 + (y_12*y_23 - y_22*y_13)^2)) == **0**

As seen, this approach for calculating the cross product of two pure or complex vectors in a space with any number of dimensions, becomes consistent. On the other hand, we can reduce as above, a resultant vector, to a real and imaginary two-term vector:

$$\mathbf{c}=\mathbf{r_1}\times\mathbf{r_2}=a\mathbf{u}_{\boldsymbol{R}}+b\hat{\boldsymbol{u}}_{\boldsymbol{I}}\qquad ...(56)$$

For:

$$\left\{\begin{aligned}a\mathbf{u}_{\boldsymbol{R}}&=\left(x_{11}x_{22}-x_{21}x_{12}\right)\mathbf{u}_{\mathbf{ij}}-\left(x_{11}x_{23}-x_{21}x_{13}\right)\mathbf{u}_{\mathbf{ik}}+\left(x_{12}x_{23}-x_{22}x_{13}\right)\mathbf{u}_{\mathbf{jk}}+\left(y_{11}y_{22}-y_{21}y_{12}\right)\mathbf{u}_{\hat{\imath}\hat{\jmath}}-\left(y_{11}y_{23}-y_{21}y_{13}\right)\mathbf{u}_{\hat{\imath}\hat{k}}+\left(y_{12}y_{23}-y_{22}y_{13}\right)\mathbf{u}_{\hat{\jmath}\hat{k}}\\b\hat{u}_I&=\left(x_{11}y_{21}-x_{21}y_{11}\right)\mathbf{u}_{\mathbf{i}\hat{\imath}}-\left(x_{11}y_{22}-x_{21}y_{12}\right)\mathbf{u}_{\mathbf{i}\hat{\jmath}}+\left(x_{11}y_{23}-x_{21}y_{13}\right)\mathbf{u}_{\mathbf{i}\hat{k}}-\left(x_{12}y_{21}-x_{22}y_{11}\right)\mathbf{u}_{\mathbf{j}\hat{\imath}}+\left(x_{12}y_{22}-x_{22}y_{12}\right)\mathbf{u}_{\mathbf{j}\hat{\jmath}}-\left(x_{12}y_{23}-x_{22}y_{13}\right)\mathbf{u}_{\mathbf{j}\hat{k}}\\&\quad+\left(x_{13}y_{21}-x_{23}y_{11}\right)\mathbf{u}_{\mathbf{k}\hat{\imath}}-\left(x_{13}y_{22}-x_{23}y_{12}\right)\mathbf{u}_{\mathbf{k}\hat{\jmath}}+\left(x_{13}y_{23}-x_{23}y_{13}\right)\mathbf{u}_{\mathbf{k}\hat{k}}\end{aligned}\right\}\qquad ...(57)$$

If it had been the case of pure vectors, the imaginary axes, $\hat{\imath},\,\hat{\jmath},\,\hat{k}$ would be null. And, the expression for vector **c**, simplifies to the known result for two 3D vectors:

$$\mathbf{c}=\mathbf{r_1}\times\mathbf{r_2}=\begin{vmatrix}x_{11}&x_{12}\\x_{21}&x_{22}\end{vmatrix}\mathbf{u}_{]\mathbf{ij}[}-\begin{vmatrix}x_{11}&x_{13}\\x_{21}&x_{23}\end{vmatrix}\mathbf{u}_{]\mathbf{ik}[}+\begin{vmatrix}x_{12}&x_{13}\\x_{22}&y_{23}\end{vmatrix}\mathbf{u}_{]\mathbf{jk}[}\qquad ...(58)$$

$$\mathbf{c} = \mathbf{r_1} \times \mathbf{r_2} = \sqrt{\left(x_{11}x_{22} - x_{21}x_{12}\right)^2 + \left(x_{11}x_{23} - x_{21}x_{13}\right)^2 + \left(x_{12}x_{23} - x_{22}x_{13}\right)^2}\ \mathbf{n} \quad ...(59)$$

In fact, for new vectors without their imaginary parts, $\mathbf{r}_1 = (x_{11})\mathbf{i} + (x_{12})\mathbf{j} + (x_{13})\mathbf{k}$, $\mathbf{r}_2 = (x_{21})\mathbf{i} + (x_{22})\mathbf{j} + (x_{23})\mathbf{k}$ and for $\mathbf{u}_{]\mathbf{ij}[} = \mathbf{k}$, $\mathbf{u}_{]\mathbf{ki}[} = \mathbf{j}$, $\mathbf{u}_{]\mathbf{jk}[} = \mathbf{i}$, by substituting and applying related changes in ascending order, we obtain the cross product we were used to:

$$\mathbf{c} = \mathbf{r_1} \times \mathbf{r_2} = \begin{vmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{vmatrix} \mathbf{u}_{]\mathbf{ij}[} - \begin{vmatrix} x_{11} & x_{13} \\ x_{21} & x_{23} \end{vmatrix} \mathbf{u}_{]\mathbf{ik}[} + \begin{vmatrix} x_{12} & x_{13} \\ x_{22} & x_{23} \end{vmatrix} \mathbf{u}_{]\mathbf{jk}[} = \begin{vmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{vmatrix} \mathbf{k} + \begin{vmatrix} x_{13} & x_{11} \\ x_{23} & x_{21} \end{vmatrix} \mathbf{j} + \begin{vmatrix} x_{12} & x_{13} \\ x_{22} & x_{23} \end{vmatrix} \mathbf{i}$$

$$\mathbf{c} = \mathbf{r_1} \times \mathbf{r_2} == \left(x_{11}x_{22} - x_{21}x_{12}\right)\mathbf{k} + \left(x_{13}x_{21} - x_{23}x_{11}\right)\mathbf{j} + \left(x_{12}x_{23} - x_{22}x_{13}\right)\mathbf{i} \quad ...(60)$$

$$\mathbf{c} = \mathbf{r_1} \times \mathbf{r_2} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \end{vmatrix} = \left(x_{12}x_{23} - x_{22}x_{13}\right)\mathbf{i} + \left(x_{13}x_{21} - x_{23}x_{11}\right)\mathbf{j} + \left(x_{11}x_{22} - x_{21}x_{12}\right)\mathbf{k}$$

$$\mathbf{c} = \mathbf{r_1} \times \mathbf{r_2} = \sqrt{\left(x_{12}x_{23} - x_{22}x_{13}\right)^2 + \left(x_{13}x_{21} - x_{23}x_{11}\right)^2 + \left(x_{11}x_{22} - x_{21}x_{12}\right)^2}\ \mathbf{n} \quad ...(61)$$

Where, as said, **n** is a vector perpendicular to $\mathbf{r}_1$ and $\mathbf{r}_2$, inside the three-dimensional space. See that Equation (61) is equal to Equation (59). With these results, we can define the general Dot and Cross product.

5.3 Dot and Cross Product of TWO Complex Vectors in a Space of "N" Dimensions

This general case actually has *2N* dimensions, *N* real and *N* imaginary ones, and originates *2N* axes and *2N* terms in the DOT product:

$$\mathbf{r_1} \cdot \mathbf{r_2} = \sum\nolimits_1^N \left(x_{1i}x_{2i} + y_{1i}y_{2i}\right) \quad ...(62)$$

And achieving $C_{2N,2} = N(2N-1)$ terms in the CROSS product, according to the arrangement:

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} & \cdots & \mathbf{n} & \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} & \cdots & \hat{\boldsymbol{n}} \\ x_{11} & x_{12} & x_{13} & \cdots & x_{1N} & y_{11} & y_{12} & y_{13} & \cdots & y_{1N} \\ x_{21} & x_{22} & x_{21} & \cdots & x_{2N} & y_{21} & y_{22} & y_{23} & \cdots & y_{2N} \end{vmatrix} \quad ...(63)$$

By constructing $N(2N-1)$ terms all of them with 2x2 matrices of the following types:

$$\begin{vmatrix} x_{1a} & x_{2b} \\ x_{2a} & x_{2b} \end{vmatrix} \mathbf{u}_{]\mathbf{ab}[}\ for \begin{Bmatrix} \mathbf{a} = \mathbf{i}\cdots\mathbf{n},\ and\ \mathbf{b} = \mathbf{j}\cdots\mathbf{n} \\ a = 1\cdots N,\ and\ b = 2\cdots N \end{Bmatrix};\ \text{or}$$

$$\begin{vmatrix} x_{1a} & y_{1\hat{d}} \\ x_{2a} & y_{2\hat{d}} \end{vmatrix} \mathbf{u}_{]\mathbf{a}\hat{d}[}\ for \begin{Bmatrix} \mathbf{a} = \mathbf{i}\cdots\mathbf{n},\ and\ \hat{\boldsymbol{d}} = \hat{\boldsymbol{i}}\cdots\hat{\boldsymbol{n}} \\ a = 1\cdots N,\ and\ \hat{d} = 2\cdots N \end{Bmatrix};$$

$$\text{or} \begin{vmatrix} y_{1\hat{c}} & y_{1\hat{d}} \\ y_{2\hat{c}} & y_{2\hat{d}} \end{vmatrix} u_{]\hat{c}\hat{d}[}\ for \begin{Bmatrix} \hat{\boldsymbol{c}} = \hat{\boldsymbol{i}}, \cdots, \hat{\boldsymbol{n}},\ and\ \hat{\boldsymbol{d}} = \hat{\boldsymbol{j}}, \cdots, \hat{\boldsymbol{n}} \\ \hat{c} = 1, \cdots, N,\ and\ \hat{d} = 2, \cdots, N \end{Bmatrix} \quad ...(64)$$

The first part of the matrix model is built starting with that involving the first column with the second $(x_{1a}x_{2b} - x_{2a}x_{2a})$, then the same first column followed by the third, and so on until the column *N* is reached (always in ascending order). Next, for the second part is repeated starting with column 2 followed by the third and then the same second column followed by the fourth and so on until the columns *N*–1 and *N* are reached, ending the real matrices, and we are at 1/3 of the procedure. Then the process continues in the same way, continuing with the first real column 1 followed by the first column of the imaginary ones (*N*+1), but with the imaginary ones named, instead of $x_{N+1,a}$ as $y_{1,a}$ until the penultimate column is reached followed by the last column, and then we are arrived at 2/3 of the procedure. The last part, only imaginary, is similar to first part (only real) but instead we will have terms as $(y_{1c}y_{2d} - y_{2c}y_{1d})$.

After constructing the matrices, and obtaining the terms of the form $(x_{1a}x_{2b} - x_{2a}x_{1b})\mathbf{u}_{|\mathbf{ab}|}$, or $(x_{1a}y_{2c} - x_{2a}y_{1c})\mathbf{u}_{|\mathbf{ac}|}$, or $(y_{1c}y_{2d} - y_{2c}y_{1d})\mathbf{u}_{|\mathbf{cd}|}$, where its subindexes refers to the position of the specified column of the real or imaginary axes, and the subindex of the unit vectors refers to the involved product of the basis vectors. Having achieved these terms, the Cross Product can be evaluated as shown for "one" (two), "two" (four) and "N" (2N) dimensions.

6. Discussion

The interpretation that the product of a real basis vector, **i**, **j**, **k**, … by the imaginary particle *j* as another basis vector, $j\mathbf{i} = \hat{\imath}$, $j\mathbf{j} = \hat{\jmath}$, $j\mathbf{k} = \hat{k}$, ..., imaginary and perpendicular to that involved, allowed us to obtain the definition of a complex vector (a complex line), with the characteristics of an authentic vector, with a real and an imaginary vector part. This new definition naturally explains the perpendicularity of the imaginary component with the real one in an authentic vector notation. Such interpretation, allowed us to calculate the scalar and cross products without the influence of the imaginary particle $j = \sqrt{-1}$, which *disappears* within the definition of the imaginary basis vectors, $\hat{\imath}, \hat{\jmath}, \hat{k}, \ldots$. It also allowed obtaining the magnitude, or modulus, of a complex vector via a dot product in the same way as it is for a pure vector: $|\mathbf{r}| = \sqrt{\mathbf{r}\cdot\mathbf{r}} = \sqrt{r^2} = r$. Viewing two complex vectors as two complex lines which have an angle θ between them $(\mathbf{r_1}\cdot\mathbf{r_2} = r_1 r_2 \cos\theta, \mathbf{r_1}\times\mathbf{r_2} = r_1 r_2 \sin\theta\,\mathbf{n})$ in a space of any number of dimensions, with the definition of imaginary vectors, without the particle *j*, is a key aspect not conceived in the works of Hamilton (1844) and of Gibbs (1884) 40 years later, in which the particle *j* was preserved within the complex number concept, forcing "its module" to be expressed as known through the scalar product of a complex number times its "conjugate": $|z| = \sqrt{z\cdot\bar{z}}$. So, the definition of the complex vector module, done by these authors in this way, introduced an unsolvable contradiction with the known definition of the module of a pure vector, $|\mathbf{r}| = \sqrt{\mathbf{r}\times\mathbf{r}}$. However, after the pioneering works of these authors on complex vectors, carriers of this misinterpretation, all those who approached this question using different paths in their investigations (Silagadze, 2002; Mcloughlin., 2013; Chengshen *et al.,* 2022), through the Jacobi identity with the application of different multidimensional matrices, or other paths, tried to reach consistent results, again in our opinion, without success. Perhaps, this could have been the main reason for not succeeding with the generalization of the scalar and cross product of two multidimensional complex vectors, in spite of the attempts made until the present years. The new interpretation given to complex vectors in this work, by using the general expressions of dot and cross product $(\mathbf{r_1}\times\mathbf{r_2} = r_1 r_2\cos\theta, \mathbf{r_1}\times\mathbf{r_2} = r_1 r_2 \sin\theta)$, to verify our view of the general used matrix model, made possible to get natural expressions of the complex vector modulus and the generalization of scalar and cross products for two vectors in a multidimensional space.

7. N-Dimensional Cross Triple-Product of Three Vectors: *c* x (*a* x *b*)

Applying the general relations given by Gibbs (1884) we could develop the triple cross product of three vectors (pure or complex). See some examples using the general relationship given in Gibbs (1884):

$$\mathbf{d} = \mathbf{c}\times(\mathbf{a}\times\mathbf{b}) = (\mathbf{c}\cdot\mathbf{b})\mathbf{a} - (\mathbf{c}\cdot\mathbf{a})\mathbf{b}$$

7.1. Three-Dimensional Cross Triple-Product of Pure Vectors: c x (a x b)

In this example, vectors **a**, **b** and ***c*** are in the plane *XYZ*. Thus, we can set $\mathbf{a} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$, $\mathbf{b} = b_x\mathbf{i} + b_y\mathbf{j} + b_z\mathbf{k}$ and $\mathbf{c} = c_x\mathbf{i} + c_y\mathbf{j} + c_z\mathbf{k}$, where, $\mathbf{d} = \mathbf{c}\times(\mathbf{a}\times\mathbf{b}) = (\mathbf{c}\cdot\mathbf{b})\mathbf{a} - (\mathbf{c}\cdot\mathbf{a})\mathbf{b}$:

On one hand,

$$\mathbf{d} = \mathbf{c} \text{ x } (\mathbf{a} \text{ x } \mathbf{b}) = (c_x b_x + c_y b_y + c_z b_z)\,(a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}) - (c_x a_x + c_y a_y + c_z a_z)\,(b_x\mathbf{i} + b_y\mathbf{j} + b_z\mathbf{k})$$

:

$$\mathbf{d} = (a_x c_x b_x + a_x c_y b_y + a_x c_z b_z - b_x c_x a_x - b_x c_y a_y - b_x c_z a_z)\mathbf{i} + (a_y c_x b_x + a_y c_y b_y + a_y c_z b_z - b_y c_x a_x - b_y c_y a_y - b_y c_z a_z)\mathbf{j} + (a_z c_x b_x + a_z c_y b_y + a_z c_z b_z - b_z c_x a_x - b_z c_y a_y - b_z c_z a_z)\mathbf{k}$$

$$\mathbf{d} = (a_x c_y b_y + a_x c_z b_z - b_x c_y a_y - b_x c_z a_z)\mathbf{i} + (a_y c_x b_x + a_y c_z b_z - b_y c_x a_x - b_y c_z a_z)\mathbf{j} + (a_z c_x b_x + a_z c_y b_y - b_z c_x a_x - b_z c_y a_y)\mathbf{k}$$

Reordering $\mathbf{d} = \mathbf{c}$ x $(\mathbf{a}$ x $\mathbf{b})$ in 2x2 matrices in ascending order, using the anticommutative law:

$$\mathbf{d} = \left(c_y\left(a_x b_y - b_x a_y\right) + c_z\left(a_x b_z - b_x a_z\right)\right)\mathbf{i} + \left(-c_x\left(z_x b_y - b_x a_y\right) + c_z\left(a_y b_z - b_y a_z\right)\right)\mathbf{j} + \left(-c_x\left(a_x b_z - b_x a_z\right) - c_y\left(a_y b_z - b_y a_z\right)\right)\mathbf{k}$$

$$= \left(c_y\begin{vmatrix} a_x & a_y \\ b_x & b_y \end{vmatrix} + c_z\begin{vmatrix} a_x & a_z \\ b_x & b_z \end{vmatrix}\right)\mathbf{i} + \left(-c_x\begin{vmatrix} a_x & a_y \\ b_x & b_y \end{vmatrix} + c_z\begin{vmatrix} a_y & a_z \\ b_y & b_z \end{vmatrix}\right)\mathbf{j} + \left(-c_x\begin{vmatrix} a_x & a_z \\ b_x & b_z \end{vmatrix} - c_y\begin{vmatrix} a_y & a_z \\ b_y & b_z \end{vmatrix}\right)\mathbf{k} \quad \text{...(65)}$$

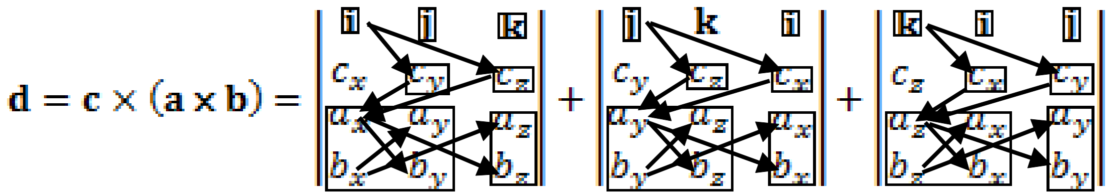


Figure 2: Graph Showing the Way to Obtain the Expression Below

On the other hand,

$$\mathbf{d}=\mathbf{c}\times(\mathbf{a}\times\mathbf{b})=(c_x\mathbf{i}+c_y\mathbf{j}+c_z\mathbf{k})\times\left\{\begin{vmatrix}a_x & a_y\\ b_x & b_y\end{vmatrix}\mathbf{u}_k-\begin{vmatrix}a_x & a_z\\ b_x & b_z\end{vmatrix}\mathbf{u}_j+\begin{vmatrix}a_y & a_z\\ b_y & b_z\end{vmatrix}\mathbf{u}_i\right\}$$

$$\mathbf{d}=c_y(a_xb_y-b_xa_y)\mathbf{u}_{jk}-c_z(a_xb_z-b_xa_z)\mathbf{u}_{kj}+c_x(a_xb_y-b_xa_y)\mathbf{u}_{ik}+c_z(a_yb_z-b_ya_z)\mathbf{u}_{ki}-c_x(a_xb_z-b_xa_z)\mathbf{u}_{ij}+c_y(a_yb_z-b_ya_z)\mathbf{u}_{ji}$$

By using $\mathbf{u}_{jk}=\mathbf{i}$, $\mathbf{u}_{ki}=\mathbf{j}$, $\mathbf{u}_{ij}=\mathbf{k}$; and changing signs according to the anticommutative law:

$$\mathbf{d}=[c_y(a_xb_y-b_xa_y)+c_z(a_xb_z-b_xa_z)]\mathbf{u}_{jk}+[-c_x(a_xb_y-b_xa_y)+c_z(a_yb_z-b_ya_z)]\mathbf{u}_{ki}+[-c_x(a_xb_z-b_xa_z)-c_y(a_yb_z-b_ya_z)]\mathbf{u}_{ij}$$

$$\mathbf{d}=\left[c_y\begin{vmatrix}a_x & a_y\\ b_x & b_y\end{vmatrix}+c_z\begin{vmatrix}a_x & a_z\\ b_x & b_z\end{vmatrix}\right]\mathbf{i}+\left[-c_x\begin{vmatrix}a_x & a_y\\ b_x & b_y\end{vmatrix}+c_z\begin{vmatrix}a_y & a_z\\ b_y & b_z\end{vmatrix}\right]\mathbf{j}+\left[-c_y\begin{vmatrix}a_x & a_y\\ b_x & b_y\end{vmatrix}-c_z\begin{vmatrix}a_y & a_z\\ b_y & b_z\end{vmatrix}\right]\mathbf{k} \quad ...(66)$$

Or, reordering into a 4x3 skewed cross product matrix, in ascending order, the result is seen directly in Figure 2.

$$\mathbf{d}=\mathbf{c}\text{ x }(\mathbf{a}\text{ x }\mathbf{b})=[c_y(a_xb_y-b_xa_y)+c_z(a_xb_z-b_xa_z)]\mathbf{i}+[-c_x(a_xb_y-b_xa_y)+c_z(a_yb_z-b_ya_z)]\mathbf{j}+[-c_x(a_xb_z-b_xa_z)$$
$$-c_y(a_yb_z-b_ya_z)]\mathbf{k} \quad ...(67)$$

Results are consistent, as in fact they must be, showing the suitability of the used models for 3D cross triple product case (but they are applicable to any multi-dimensional cross triple product).

7.2. Four-Dimensional Cross Product of Three Pure Vectors

$$\mathbf{d}=\mathbf{c}\times(\mathbf{a}\times\mathbf{b})=(\mathbf{c}\cdot\mathbf{b})\mathbf{a}-(\mathbf{c}\cdot\mathbf{a})\mathbf{b} \quad ...(68)$$

Setting $\mathbf{a}=a_x\mathbf{i}+a_y\mathbf{j}+a_v\mathbf{v}+a_w\mathbf{w}$, $\mathbf{b}=b_x\mathbf{i}+b_y\mathbf{j}+b_v\mathbf{v}+b_w\mathbf{w}$ and $\mathbf{c}=c_x\mathbf{i}+c_y\mathbf{j}+c_v\mathbf{v}+c_w\mathbf{w}$.

Using, $\mathbf{c}\cdot\mathbf{b}=c_xb_x+c_yb_y+c_vb_v+c_wb_w$ and $\mathbf{c}\cdot\mathbf{a}=c_xa_x+c_ya_y+c_va_v+c_wa_w$, on the one hand, by developing $(\mathbf{c}\cdot\mathbf{b})\mathbf{a}-(\mathbf{c}\cdot\mathbf{a})\mathbf{b}$ for $\mathbf{d}=\mathbf{c}\text{ x }(\mathbf{a}\text{ x }\mathbf{b})=(\mathbf{c}\cdot\mathbf{b})\mathbf{a}-(\mathbf{c}\cdot\mathbf{a})\mathbf{b}$, we have:

$$\mathbf{d}=(c_xb_x+c_yb_y+c_vb_v+c_wb_w)(a_x\mathbf{i}+a_y\mathbf{j}+a_v\mathbf{v}+a_w\mathbf{w})-(c_xa_x+c_ya_y+c_va_v+c_wa_w)(b_x\mathbf{i}+b_y\mathbf{j}+b_v\mathbf{v}+b_w\mathbf{w})$$

$$\mathbf{d}=(c_y(a_xb_y-b_xa_y)+c_v(a_xb_v-b_xa_v)+c_w(a_xb_w-b_xa_w))\mathbf{i}+(c_x(a_yb_x-b_ya_x)+c_v(a_yb_v-b_ya_v)+c_w(a_yb_w-b_ya_w))\mathbf{j}+(c_x(a_vb_x$$
$$-b_va_x)+c_y(a_vb_y-b_va_y)+c_w(a_vb_w-b_va_w))\mathbf{v}+(c_x(a_wb_x-b_wa_x)+c_y(a_wb_y-b_wa_y)+c_v(a_wb_v-b_wa_v))\mathbf{w} \quad ...(69)$$

In this result, by changing the 2x2 matrices with their subscripts to those of the ascending order, and using the law of anti-commutativity, **d** changes to:

$$\mathbf{d}=(c_y(a_xb_y-b_xa_y)+c_v(a_xb_v-b_xa_v)+c_w(a_xb_w-b_xa_w))\mathbf{i}+(-c_x(a_xb_y-b_xa_y)+c_v(a_yb_v-b_ya_v)+c_w(a_yb_w-b_ya_w))\mathbf{j}+(-$$
$$c_x(a_xb_v-b_xa_v)-c_y(a_yb_v-b_ya_v)+c_w(a_vb_w-b_va_w))\mathbf{v}+(-c_x(a_xb_w-b_xa_w)-c_y(a_yb_w-b_ya_w)-c_v(a_vb_w-b_va_w))\mathbf{w} \quad ...(70)$$

On the other hand, (**a** x **b**) expressed as 2x2 matrices (total: $C_{4,2}=6$), and since each 2x2 matrix comes from eliminating two columns and the first row, leaving bivector unit bases, we can put:

$$\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{v} & \mathbf{w}\\ a_x & a_y & a_v & a_w\\ b_x & b_y & b_v & b_w\end{vmatrix}\text{, with bivectors defined as }\left\{\begin{matrix}\mathbf{u}_{vw}=\mathbf{v},\mathbf{w} & \mathbf{u}_{jw}=-\mathbf{j},\mathbf{w}\\ \mathbf{u}_{jv}=-\mathbf{j},-\mathbf{v} & \mathbf{u}_{iw}=\mathbf{i},\mathbf{w}\\ \mathbf{u}_{iv}=\mathbf{i},-\mathbf{v} & \mathbf{u}_{ij}=\mathbf{i},\mathbf{j}\end{matrix}\right\}\text{ then:}$$

$$\mathbf{a}\times\mathbf{b}=\pm\begin{vmatrix}a_x & a_y\\ b_x & b_y\end{vmatrix}\mathbf{u}_{vw}\pm\begin{vmatrix}a_x & a_v\\ b_x & b_v\end{vmatrix}\mathbf{u}_{jw}\pm\begin{vmatrix}a_x & a_w\\ b_x & b_w\end{vmatrix}\mathbf{u}_{jv}\pm\begin{vmatrix}a_y & a_v\\ b_y & b_v\end{vmatrix}\mathbf{u}_{iw}\pm\begin{vmatrix}a_y & a_w\\ b_y & b_w\end{vmatrix}\mathbf{u}_{iv}\pm\begin{vmatrix}a_v & a_w\\ b_v & b_w\end{vmatrix}\mathbf{u}_{ij} \qquad ...(71)$$

The sign of c_p in **c** x (**a** x **b**) in this 4D case may be positive or negative, depending on its use. Let's start with the ± sign for all terms to realize how the law of formation of signs arises:

$$\mathbf{d}=\mathbf{c}\times(\mathbf{a}\times\mathbf{b})=(c_x\mathbf{i}+c_y\mathbf{j}+c_v\mathbf{v}+c_w\mathbf{w})$$

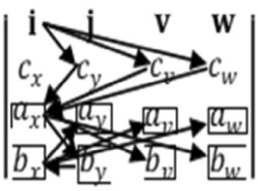
$$\times\left\{\pm\begin{vmatrix}a_x & a_y\\ b_x & b_y\end{vmatrix}\mathbf{u}_{vw}\pm\begin{vmatrix}a_x & a_v\\ b_x & b_v\end{vmatrix}\mathbf{u}_{jw}\pm\begin{vmatrix}a_x & a_w\\ b_x & b_w\end{vmatrix}\mathbf{u}_{jv}\pm\begin{vmatrix}a_y & a_v\\ b_y & b_v\end{vmatrix}\mathbf{u}_{iw}\pm\begin{vmatrix}a_y & a_w\\ b_y & b_w\end{vmatrix}\mathbf{u}_{iv}\pm\begin{vmatrix}a_v & a_w\\ b_v & b_w\end{vmatrix}\mathbf{u}_{ij}\right\}$$

$$\mathbf{d}=\pm c_y(a_xb_y-b_xa_y)\mathbf{u}_{jvw}\pm c_v(a_xb_v-b_xa_v)\mathbf{u}_{vjw}\pm c_w(a_xb_w-b_xa_w)\mathbf{u}_{wjv}\pm c_x(a_xb_y-b_xa_y)\mathbf{u}_{ivw}\pm c_v(a_yb_v-b_ya_v)\mathbf{u}_{viw}\pm c_w(a_yb_w-b_ya_w)\mathbf{u}_{wiv}\pm c_x(a_xb_v-b_xa_v)\mathbf{u}_{ijw}\pm c_y(a_yb_v-b_ya_v)\mathbf{u}_{jiw}\pm c_w(a_vb_w-b_va_w)\mathbf{u}_{wij}\pm c_x(a_xb_w-b_xa_w)\mathbf{u}_{ijv}\pm c_y(a_yb_w-b_ya_w)\mathbf{u}_{jiv}\pm c_v(a_vb_w-b_va_w)\mathbf{u}_{vij}$$

By taking the unit bases with three-letter subscripts as equal to that missing, i.e.: $\mathbf{u}_{jvw}=\mathbf{j}\times(\mathbf{v},\mathbf{w})=\mathbf{i}$, $\mathbf{u}_{vwi}=\mathbf{v}\times(\mathbf{w},\mathbf{i})=\mathbf{j}$, $\mathbf{u}_{wij}=\mathbf{w}\times(\mathbf{i},\mathbf{j})=\mathbf{v}$, $\mathbf{u}_{ijv}=\mathbf{i}\times(\mathbf{j},\mathbf{v})=\mathbf{w}$; and giving negative sign to the matrix without the linked missing basis $\boldsymbol{m}(\mathbf{i},\mathbf{j},\mathbf{v},\mathbf{w})$ in $\boldsymbol{a}_m(\boldsymbol{a}_{x,y,v,w})$, inside the parenthesis, as the sign of the coefficient c_p, (bold = positive) for $p = x, y, v, w$, we obtain:

$$\mathbf{d}=[c_y(\boldsymbol{a}_xb_y-b_xa_y)+c_v(\boldsymbol{a}_xb_v-b_xa_v)+c_w(\boldsymbol{a}_xb_w-b_xa_w)]\mathbf{i}+[-c_x(a_xb_y-b_x\boldsymbol{a}_y)+c_v(\boldsymbol{a}_yb_v-b_ya_v)+c_w(\boldsymbol{a}_yb_w-b_ya_w)]\mathbf{j}+[-c_x(a_xb_v-b_xa_v)-c_y(a_yb_v-b_ya_v)+c_w(\boldsymbol{a}_vb_w-b_va_w)]\mathbf{v}+[-c_x(a_xb_w-b_xa_w)-c_y(a_yb_w-b_ya_w)-c_v(a_vb_w-b_va_w)]\mathbf{w} \qquad ...(72)$$

Note: This process can be done directly by operating the following ascending order 4x4 matrix and locating the first column as the last one after each matrix operation, and doing the 2x2 last rows as indicated by the skewed arrows until finishing:



$$=\left\{\begin{aligned}&\mathbf{i}[c_y(\boldsymbol{a}_xb_y-b_xa_y)+c_v(\boldsymbol{a}_xb_v-b_xa_v)+c_w(\boldsymbol{a}_xb_w-b_xa_w)]\\&+\mathbf{j}[-c_x(a_xb_y-b_xa_y)+c_v(\boldsymbol{a}_yb_v-b_ya_v)+c_w(\boldsymbol{a}_yb_w-b_ya_w)]\\&+\mathbf{v}[-c_x(a_xb_v-b_xa_v)-c_y(a_yb_v-b_ya_v)+c_w(\boldsymbol{a}_vb_w-b_va_w)]\\&+\mathbf{w}[-c_x(a_xb_w-b_xa_w)-c_y(a_yb_w-b_ya_w)-c_v(a_vb_w-b_va_w)]\end{aligned}\right\}$$

Figure 3: Graph Showing the Way to Obtain the Expression on the Right

So, we arrive at a replica of the results found above. This procedure to obtain the sign is easily generalizable and applicable to multidimensional cross triple-product of complex vectors.

7.3. Fifth-Dimensional Cross Product of Three Pure Vectors: c x (a x b)

Setting $\mathbf{a}=a_x\mathbf{i}+a_y\mathbf{j}+a_v\mathbf{v}+a_w\mathbf{w}+a_k\mathbf{k}$, $\mathbf{b}=b_x\mathbf{i}+b_y\mathbf{j}+b_v\mathbf{v}+b_w\mathbf{w}+b_k\mathbf{k}$ and $\mathbf{c}=c_x\mathbf{i}+c_y\mathbf{j}+c_v\mathbf{v}+c_w\mathbf{w}+c_k\mathbf{k}$, and developing **a** x **b** as a sum, $C_{5,2}=10$, of 2 x 2 matrices in ascending order and using, $\mathbf{c}\cdot\mathbf{b}=c_xb_x+c_yb_y+c_vb_v+c_wb_w+c_kb_k$ and $\mathbf{c}\cdot\mathbf{a}=c_xa_x+c_ya_y+c_va_v+c_wa_w+c_ka_k$ on the one hand, we have for $\mathbf{d}=\mathbf{c}\times(\mathbf{a}\times\mathbf{b})=(\mathbf{c}\cdot\mathbf{b})\mathbf{a}-(\mathbf{c}\cdot\mathbf{a})\mathbf{b}$:

$$\mathbf{d}=(\mathbf{c}\cdot\mathbf{b})\mathbf{a}-(\mathbf{c}\cdot\mathbf{a})\mathbf{b}$$

$$\mathbf{d}=(\mathbf{c}\cdot\mathbf{b})\,(a_x\mathbf{i}+a_y\mathbf{j}+a_v\mathbf{v}+a_w\mathbf{w}+a_k\mathbf{k})-(\mathbf{c}\cdot\mathbf{a})\,(b_x\mathbf{i}+b_y\mathbf{j}+b_v\mathbf{v}+b_w\mathbf{w}+b_k\mathbf{k})$$

$$\mathbf{d}=(c_y(a_xb_y-b_xa_y)+c_v(a_xb_v-b_xa_v)+c_w(a_xb_w-b_xa_w)+c_k(a_xb_k-b_xa_k))\mathbf{i}+(c_x(a_yb_x-b_ya_x)+c_v(a_yb_v-b_ya_v)+c_w(a_yb_w-b_ya_w)+c_k(a_yb_k-b_ya_k))\mathbf{j}+(c_x(a_vb_x-b_va_x)+c_y(a_vb_y-b_va_y)+c_w(a_vb_w-b_va_w)+c_k(a_vb_k-b_va_k))\mathbf{v}+(c_x(a_wb_x-b_wa_x)+c_y(a_wb_y-b_wa_y)+c_v(a_wb_v-b_wa_v)+c_k(a_wb_k-b_wa_k))\mathbf{w}+(c_x(a_kb_y-b_ka_y)+c_y(a_kb_y-b_ka_y)+c_v(a_kb_v-b_xa_v)+c_w(a_kb_w-b_ka_w))\mathbf{i}$$

$$\mathbf{d}=(c_y(a_xb_y-b_xa_y)+c_v(a_xb_v-b_xa_v)+c_w(a_xb_w-b_xa_w)+c_k(a_xb_k-b_xa_k))\mathbf{i}+(-c_x(a_xb_y-b_xa_y)+c_v(a_yb_v-b_ya_v)+c_w(a_yb_w-b_ya_w)+c_k(a_xb_k-b_xa_k))\mathbf{j}+(-c_x(a_xb_v-b_xa_v)-c_y(a_yb_v-b_ya_v)+c_w(a_vb_w-b_va_w)+c_k(a_xb_k-b_xa_k))\mathbf{v}+(-c_x(a_xb_w-b_xa_w)-c_y(a_yb_w-b_ya_w)-c_v(a_vb_w-b_va_w)+c_k(a_xb_k-b_xa_k))\mathbf{w}+(-c_x(a_xb_k-b_xa_k)-c_y(a_yb_k-b_ya_k)-c_v(a_vb_k-b_va_k)-c_w(a_wb_k-b_wa_k))\mathbf{k} \qquad ...(73)$$

On the other hand, the 3x5 matrix (**a** x **b**) expressed as $C_{5,2} = 10$ matrices (2x2) becomes:

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{v} & \mathbf{w} & \mathbf{k} \\ a_x & a_y & a_v & a_w & a_k \\ b_x & b_y & b_v & b_w & b_k \end{vmatrix} = \pm\begin{vmatrix} a_x & a_y \\ b_x & b_y \end{vmatrix}\mathbf{u}_{vwk} \pm \begin{vmatrix} a_x & a_v \\ b_x & b_v \end{vmatrix}\mathbf{u}_{jwk} \pm \begin{vmatrix} a_x & a_w \\ b_x & b_w \end{vmatrix}\mathbf{u}_{jvk} \pm \begin{vmatrix} a_x & a_k \\ b_x & b_k \end{vmatrix}\mathbf{u}_{jvw} \pm \begin{vmatrix} a_y & a_v \\ b_y & b_v \end{vmatrix}\mathbf{u}_{iwk} \pm \begin{vmatrix} a_y & a_w \\ b_y & b_w \end{vmatrix}\mathbf{u}_{ivk}$$

$$\pm\begin{vmatrix} a_y & a_k \\ b_y & b_k \end{vmatrix}\mathbf{u}_{ivw} \pm \begin{vmatrix} a_v & a_w \\ b_v & b_w \end{vmatrix}\mathbf{u}_{ijk} \pm \begin{vmatrix} a_v & a_k \\ b_v & b_k \end{vmatrix}\mathbf{u}_{ijw} \pm \begin{vmatrix} a_w & a_k \\ b_w & b_k \end{vmatrix}\mathbf{u}_{ijv}$$

Note: Each 2x2 matrix comes from eliminating three columns and the first row, containing three-vector unit bases. The sign of each matrix, could be positive or negative, depending on its use. Let's start with positive sign to all matrices to check the law of formation of signs:

$$\mathbf{d} = \mathbf{c}\times(\mathbf{a}\times\mathbf{b}) = (c_x\mathbf{i} + c_y\mathbf{j} + c_v\mathbf{v} + c_w\mathbf{w} + c_k\mathbf{k})$$

$$\times\left\{\begin{aligned} &\begin{vmatrix} a_x & a_y \\ b_x & b_y \end{vmatrix}\mathbf{u}_{vwk} + \begin{vmatrix} a_x & a_v \\ b_x & b_v \end{vmatrix}\mathbf{u}_{jwk} + \begin{vmatrix} a_x & a_w \\ b_x & b_w \end{vmatrix}\mathbf{u}_{jvk} + \begin{vmatrix} a_x & a_k \\ b_x & b_k \end{vmatrix}\mathbf{u}_{jvw} + \begin{vmatrix} a_y & a_v \\ b_y & b_v \end{vmatrix}\mathbf{u}_{iwk} \\ &+\begin{vmatrix} a_y & a_w \\ b_y & b_w \end{vmatrix}\mathbf{u}_{ivk} + \begin{vmatrix} a_y & a_k \\ b_y & b_k \end{vmatrix}\mathbf{u}_{ivw} \pm \begin{vmatrix} a_v & a_w \\ b_v & b_w \end{vmatrix}\mathbf{u}_{ijk} + \begin{vmatrix} a_v & a_k \\ b_v & b_k \end{vmatrix}\mathbf{u}_{ijw} + \begin{vmatrix} a_w & a_k \\ b_w & b_k \end{vmatrix}\mathbf{u}_{ijv} \end{aligned}\right\}$$

$= (c_y(a_xb_y - b_xa_y) + c_v(a_xb_v - b_xa_v) + c_w(a_xb_w - b_xa_w) + c_k(a_xb_k - b_xa_k))\mathbf{u}_{jvwk} + (-c_x(a_xb_y - b_xa_y) + c_v(a_yb_v - b_ya_v) + c_w(a_yb_w - b_ya_w) + c_k(a_xb_k - b_xa_k))\mathbf{u}_{vwki} + (-c_x(a_xb_v - b_xa_v) - c_y(a_yb_v - b_ya_v) + c_w(a_vb_w - b_va_w) + c_k(a_xb_k - b_xa_k))\mathbf{u}_{wkij} + (-c_x(a_xb_w - b_xa_w) - c_y(a_yb_w - b_ya_w) - c_v(a_vb_w - b_va_w) + c_k(a_xb_k - b_xa_k))\mathbf{u}_{kijv} + (-c_x(a_xb_k - b_xa_k) - c_y(a_yb_k - b_ya_k) - c_v(a_vb_k - b_va_k) - c_w(a_wb_k - b_wa_k))\mathbf{u}_{ijvw}$ ...(74)

By taking the unit bases with four-letter subscripts as equal to that missing, i.e.: $\mathbf{u}_{jvwk} = \mathbf{j}$ x (**v**, **w**, **k**) = **i**, $\mathbf{u}_{vwki} = \mathbf{v}$ x (**w**, **k**, **i**) = **j**, $\mathbf{u}_{wkij} = \mathbf{w}$ x (**k**, **i**, **j**) = **v**, $\mathbf{u}_{kijv} = \mathbf{k}$ x (**i**, **j**, **v**) = **w**, $\mathbf{u}_{ijvw} = \mathbf{i}$ x (**j**, **v**, **w**) = **k**; and giving negative sign to the matrix without the linked missing basis **m(i, j, v, w, k)** in $\boldsymbol{a}_m(\boldsymbol{a}_{x,y,v,w,k})$, inside the parenthesis, as the sign of the coefficient c_p, for $p = x, y, v, w$, we obtain:

$\mathbf{d} = (c_y(\boldsymbol{a}_{\boldsymbol{x}}b_y - b_xa_y) + c_v(\boldsymbol{a}_{\boldsymbol{x}}b_v - b_xa_v) + c_w(\boldsymbol{a}_{\boldsymbol{x}}b_w - b_xa_w) + c_k(\boldsymbol{a}_{\boldsymbol{x}}b_k - b_xa_k))\mathbf{i} + (-c_x(a_xb_y - b_xa_y) + c_v(\boldsymbol{a}_{\boldsymbol{y}}b_v - b_ya_v) + c_w(\boldsymbol{a}_{\boldsymbol{y}}b_w - b_ya_w) + c_k(\boldsymbol{a}_{\boldsymbol{y}}b_k - b_ya_k))\mathbf{j} + (-c_x(a_xb_v - b_xa_v) - c_y(a_yb_v - b_ya_v) + c_w(\boldsymbol{a}_{\boldsymbol{v}}b_w - b_va_w) + c_k(\boldsymbol{a}_{\boldsymbol{v}}b_k - b_va_k))\mathbf{v} + (-c_x(a_xb_w - b_xa_w) - c_y(a_yb_w - b_ya_w) - c_v(a_vb_w - b_va_w) + c_k(\boldsymbol{a}_{\boldsymbol{w}}b_k - b_wa_k))\mathbf{w} + (-c_x(a_xb_k - b_xa_k) - c_y(a_yb_k - b_ya_k) - c_v(a_vb_k - b_va_k) - c_w(a_wb_k - b_wa_k))\mathbf{k}$ ...(75)

This process, can be done directly by operating in ascending order the 4x5 vector matrix, moving the first column to the last one, after each matrix operation, until finishing, generalizable, easily programmable and applicable to any 4xN (or 4x2N in the case of complex vectors) vector matrix:

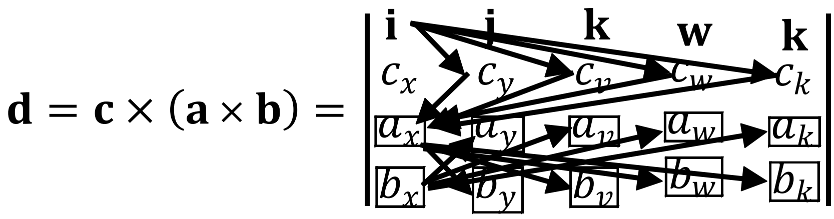


Figure 4: Graph Showing the Way to Proceed with any 4xN Matrix of Pure Vectors

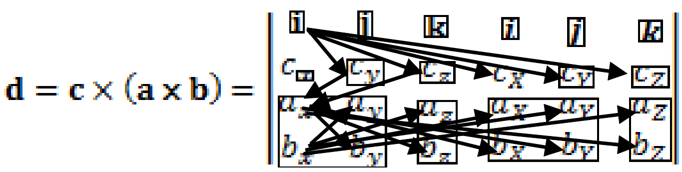


Figure 5: Graph Showing the Way to Proceed with a 4x6 Matrix of Complex Vectors

7.4. Three-Dimensional Cross Product of Complex Vectors

$\mathbf{c} \text{ x } (\mathbf{a} \text{ x } \mathbf{b}) = (\mathbf{c} \cdot \mathbf{b})\mathbf{a} - (\mathbf{c} \cdot \mathbf{a})\mathbf{b}$

The procedure used for cross triple product of pure vectors is similarly applied on 3D-complex vectors (really 6D). For: $\mathbf{a} = (a_x + ja_x)\mathbf{i} + (a_y + ja_y)\mathbf{j} + (a_z + ja_z)\mathbf{k}$, $\mathbf{b} = (b_x + jb_x)\mathbf{i} + (b_y + jb_y)\mathbf{j} + (b_z + jb_z)\mathbf{k}$, $\mathbf{c} = (c_x + jc_x)\mathbf{i} + (c_y + jc_y)\mathbf{j} + (c_z + jc_z)\mathbf{k}$. Applying as above the 4x6 skewed complex cross product matrix model the result is obtained directly:

$$\mathbf{d} = [c_y(a_xb_y - b_xa_y) + c_z(a_xb_z - b_xa_z) + c_X(a_xb_X - b_xa_X) + c_Y(a_xb_Y - b_xa_Y) + c_Z(a_xb_Z - b_xa_Z)]\mathbf{i} + [-c_x(a_xb_y - b_xa_y) + c_z(a_yb_z - b_ya_z) + c_X(a_yb_X - b_ya_X) + c_Y(a_yb_Y - b_ya_Y) + c_Z(a_yb_Z - b_ya_Z)]\mathbf{j} + [-c_x(a_xb_z - b_xa_z) - c_y(a_yb_z - b_ya_z) + c_X(a_zb_X - b_za_X) + c_Y(a_zb_y - b_za_Y) + c_Z(a_zb_Z - b_za_Z)]\mathbf{k} + [-c_x(a_xb_X - b_xa_X) - c_y(a_yb_X - b_ya_X) - c_z(a_zb_X - b_za_X) + c_Y(a_Xb_Y - b_Xa_Y) + c_Z(a_Xb_Z - b_Xa_Y)]\mathbf{i} + [-c_x(a_xb_Y - b_xa_Y) - c_y(a_yb_Y - b_ya_Y) - c_z(a_zb_Y - b_za_Y) - c_X(a_Xb_Y - b_Xa_Y) + c_Z(a_Yb_Z - b_Ya_Z)]\mathbf{j} + [-c_x(a_xb_Z - b_xa_Z) - c_y(a_yb_Z - b_ya_Z) - c_z(a_zb_Z - b_za_Z) - c_X(a_Xb_Z - b_Xa_Z) - c_Y(a_Yb_Z - b_Ya_Z)]\mathbf{k} \quad ...(76)$$

This must be equal Gibbs (1884) to $\mathbf{d} = \mathbf{c} \text{ x } (\mathbf{a} \text{ x } \mathbf{b}) = (\mathbf{c} \cdot \mathbf{b})\mathbf{a} - (\mathbf{c} \cdot \mathbf{a})\mathbf{b}$, as indeed it is.

$$\mathbf{d} = (\mathbf{c} \cdot \mathbf{b})\mathbf{a} - (\mathbf{c} \cdot \mathbf{a})\mathbf{b} = (c_ba_x + c_ab_x)\mathbf{i} + (c_ba_y + c_ab_y)\mathbf{j} + (c_ba_z + c_ab_z)\mathbf{k} + (c_ba_X + c_ab_X)\mathbf{i} + (c_ba_Y + c_ab_Y)\mathbf{j} + (c_ba_Z + c_ab_Z)\mathbf{k} \quad ...(77)$$

$$\text{For: } \begin{Bmatrix} c_a = c_xa_x + c_ya_y + c_za_z + c_Xa_X + c_Ya_Y + c_Za_Z \\ c_b = c_xb_x + c_yb_y + c_zb_z + c_Xb_X + c_Yb_Y + c_Zb_Z \end{Bmatrix} \quad ...(78)$$

Applying definitions in Equation (78) and developing the differences in Equation (77), is obtained the expression (76).

8. N-Dimensional Dot Triple-Product of Three Vectors

Let's see examples checking the relation given in Gibbs (1884):

$$\mathbf{d} = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$$

8.1. Three-Dimensional Dot Triple Product of Pure Vectors

Now, we will try to obtain a direct procedure for the Dot product. Let's start with the particular case of $\mathbf{a} = a_x\mathbf{i} + a_y\mathbf{j}$, $\mathbf{b} = b_x\mathbf{i} + b_y\mathbf{j}$ and $\mathbf{c} = c_x\mathbf{i} + c_y\mathbf{j} + c_z\mathbf{k}$.

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x & a_y & 0 \\ b_x & b_y & 0 \end{vmatrix} = \left(a_xb_y - b_xa_y\right)\mathbf{k} \, ;$$

$$\mathbf{b} \times \mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ b_x & b_y & 0 \\ c_x & c_y & c_z \end{vmatrix} = b_yc_z\mathbf{i} - b_xc_z\mathbf{j} + \left(c_ya_x - a_xc_y\right)\mathbf{k}$$

$$\mathbf{c} \times \mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ c_x & c_y & c_z \\ a_x & a_y & 0 \end{vmatrix} = -c_za_y\mathbf{i} - c_za_x\mathbf{j} + \left(c_xa_y - a_xc_y\right)\mathbf{k}$$

$$\mathbf{d} = \mathbf{c} \cdot (\mathbf{a} \text{ x } \mathbf{b}) = (c_x\mathbf{i} + c_y\mathbf{j} + c_z\mathbf{k}) \cdot (a_xb_y - b_xa_y)\mathbf{k} = c_z(a_xb_y - b_xa_y) \quad ...(79)$$

$$\mathbf{d} = \mathbf{b} \cdot (\mathbf{c} \text{ x } \mathbf{a}) = (b_x\mathbf{i} + b_y\mathbf{j}) \cdot [-c_za_y\mathbf{i} + c_za_x\mathbf{j} + (c_ya_x - a_xc_y)\mathbf{k}] = c_z(a_xb_y - b_xa_y) \quad ...(80)$$

$$\mathbf{d} = \mathbf{a} \cdot (\mathbf{b} \text{ x } \mathbf{c}) = (a_x\mathbf{i} + a_y\mathbf{j}) \cdot [b_yc_z\mathbf{i} - b_xc_z\mathbf{j} + (c_ya_x - a_xc_y)\mathbf{k}] = c_z(a_xb_y - b_xa_y) \quad ...(81)$$

In the general 3D case, for $\mathbf{a} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$, $\mathbf{b} = b_x\mathbf{i} + b_y\mathbf{j} + b_z\mathbf{k}$, and $\mathbf{c} = c_x\mathbf{i} + c_y\mathbf{j} + c_z\mathbf{k}$

$$\mathbf{a}\times\mathbf{b}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ a_x & a_y & a_z\\ b_x & b_y & b_z\end{vmatrix}=\left(a_y b_z-b_y a_z\right)\mathbf{i}-\left(a_x b_z-b_x a_z\right)\mathbf{j}+\left(a_x b_y-b_x a_y\right)\mathbf{k}$$

$$\mathbf{d}=\mathbf{c}\cdot(\mathbf{a}\ \mathrm{x}\ \mathbf{b})=(c_x\mathbf{i}+c_y\mathbf{j}+c_z\mathbf{k})\cdot[(a_y b_z-b_y a_z)\mathbf{i}-(a_x b_z-b_x a_z)\mathbf{j}+(a_x b_y-b_x a_y)\mathbf{k}]$$

$$\mathbf{d}=\mathbf{c}\cdot(\mathbf{a}\ \mathrm{x}\ \mathbf{b})=c_x(a_y b_z-b_y a_z)-c_y(a_x b_z-b_x a_z)+c_z(a_x b_y-b_x a_y)=c_x(a_y b_z-b_y a_z)+c_y(a_z b_x-b_z a_x)+c_z(a_x b_y-b_x a_y) \quad ...(82)$$

Or using directly:

$$\begin{vmatrix}c_x & c_y & c_z\\ a_x & a_y & a_z\\ b_x & b_y & b_z\end{vmatrix}=c_x\left(a_y b_z-b_y a_z\right)-c_y\left(a_x b_z-b_x a_z\right)+c_z\left(a_x b_y-b_x a_y\right) \quad ...(83)$$

This is the volume of a parallelepiped of sides **a**, **b** and **c** for the particular and general 3D-cases.

8.2. Four-Dimensional Dot Product of Three Pure Vectors

$\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})=\mathbf{b}\cdot(\mathbf{c}\times\mathbf{a})=\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$

This case should provide the volume of a three-dimensional parallelepiped in a four-dimensional space. Thus, as above when making the products $\mathbf{c}\cdot(\mathbf{a}\ \mathrm{x}\ \mathbf{b})$ with 2x2 matrices considering the subscripts of the unit basis in ascending order, positive signs are assigned if added subscript of the factor included (c_x, c_y, c_v, c_w) multiplying the parenthesis is outside the ascending order of the subscripts located inside the parenthesis. If this is not occurring, negative signs apply:

$$\mathbf{d}=\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})=\left(c_x\mathbf{i}+c_y\mathbf{j}+c_v\mathbf{v}+c_w\mathbf{w}\right)\cdot\left\{\begin{vmatrix}a_x & a_y\\ b_x & b_y\end{vmatrix}\mathbf{u}_{vw}+\begin{vmatrix}a_x & a_v\\ b_x & b_v\end{vmatrix}\mathbf{u}_{jw}+\begin{vmatrix}a_x & a_w\\ b_x & b_w\end{vmatrix}\mathbf{u}_{jv}+\begin{vmatrix}a_y & a_v\\ b_y & b_v\end{vmatrix}\mathbf{u}_{iw}+\begin{vmatrix}a_y & a_w\\ b_y & b_w\end{vmatrix}\mathbf{u}_{iv}+\begin{vmatrix}a_v & a_w\\ b_v & b_w\end{vmatrix}\mathbf{u}_{ij}\right\} \quad ...(85)$$

So, multiplying matrices by coefficients c_m indicated by subscripts of $\mathbf{u}_{\mathbf{pq}}=\mathbf{p}+\mathbf{q}$, and assigning the positive sign if m is outside the subscripts order of $\boldsymbol{u}_{pq}$ and the negative sign if not, we have:

$\mathbf{d}=\mathbf{c}\cdot(\mathbf{a}\ \mathrm{x}\ \mathbf{b})=(c_x\mathbf{i}+c_y\mathbf{j}+c_v\mathbf{v}+c_w\mathbf{w})\cdot[(a_x b_y-b_x a_y)(\mathbf{v},\mathbf{w})+(a_x b_v-b_x a_v)(-\mathbf{j},\mathbf{w})+(a_x b_w-b_x a_w)(-\mathbf{j},-\mathbf{v})+(a_y b_v-b_y a_v)(\mathbf{i},\mathbf{w})+(a_y b_w-b_y a_w)(\mathbf{i},-\mathbf{v})+(a_v b_w-b_v a_w)(\mathbf{i},\mathbf{j})]$

$\mathbf{d}=\mathbf{c}\cdot(\mathbf{a}\ \mathrm{x}\ \mathbf{b})=(c_v+c_w)(a_x b_y-b_x a_y)+(-c_y+c_w)(a_x b_v-b_x a_v)+(-c_y-c_v)(a_x b_w-b_x a_w)+(c_x+c_w)(a_y b_v-b_y a_v)+(c_x-c_v)(a_y b_w-b_y a_w)+(c_x+c_y)(a_v b_w-b_v a_w)$

$\mathbf{d}=\mathbf{c}\cdot(\mathbf{a}\ \mathrm{x}\ \mathbf{b})=c_x[(a_y b_v-b_y a_v)+(a_y b_w-b_y a_w)+(a_v b_w-b_v a_w)]+c_y[-(a_x b_w-b_x a_w)-(a_x b_v-b_x a_v)+(a_v b_w-b_v a_w)]+c_v[(a_x b_y-b_x a_y)-(a_x b_w-b_x a_w)-(a_y b_w-b_y a_w)]+c_w[(a_x b_y-b_x a_y)+(a_x b_v-b_x a_v)+(a_y b_v-b_y a_v)]$ ...(86)

Similarly,

$$\mathbf{d}=\mathbf{b}\cdot(\mathbf{c}\times\mathbf{a})=\left(b_x\mathbf{i}+b_y\mathbf{j}+b_v\mathbf{v}+b_w\mathbf{w}\right)\cdot\left\{\begin{vmatrix}c_x & c_y\\ a_x & a_y\end{vmatrix}\mathbf{u}_{vw}+\begin{vmatrix}c_x & c_v\\ a_x & a_v\end{vmatrix}\mathbf{u}_{jw}+\begin{vmatrix}c_x & c_w\\ a_x & a_w\end{vmatrix}\mathbf{u}_{jv}+\begin{vmatrix}c_y & c_v\\ a_y & a_v\end{vmatrix}\mathbf{u}_{iw}+\begin{vmatrix}c_y & c_w\\ a_y & a_w\end{vmatrix}\mathbf{u}_{iv}+\begin{vmatrix}c_v & c_w\\ a_v & a_w\end{vmatrix}\mathbf{u}_{ij}\right\}$$

$=(b_x\mathbf{i}+b_y\mathbf{j}+b_v\mathbf{v}+b_w\mathbf{w})\cdot[(c_x a_y-a_x c_y)(\mathbf{v},\mathbf{w})+(c_x a_v-a_x c_v)(-\mathbf{j},\mathbf{w})+(c_x a_w-a_x c_w)(-\mathbf{j},-\mathbf{v})+(c_y a_v-a_y c_v)(\mathbf{i},\mathbf{w})+(c_y a_w-a_y c_w)(\mathbf{i},-\mathbf{v})+(c_v a_w-a_v w_w)(\mathbf{i},\mathbf{j})]$

$=(b_v+b_w)(c_x a_y-a_x c_y)+(-b_y+b_w)(c_x a_v-a_x c_v)+(-b_y-b_v)(c_x a_w-a_x c_w)+(b_x+b_w)(c_y a_v-a_y c_v)+(b_x-b_v)(c_y a_w-a_y c_w)+(b_x+b_y)(c_v a_w-a_v c_w)$

$=c_x[(a_y b_v-b_y a_v)+(a_y b_w-b_y a_w)+(a_v b_w-b_v a_w)]+c_y[-(a_x b_w-b_x a_w)-(a_x b_v-b_x a_v)+(a_v b_w-b_v a_w)]+c_v[(a_x b_y-b_x a_y)-(a_x b_w-b_x a_w)-(a_y b_w-b_y a_w)]+c_w[(a_x b_y-b_x a_y)+(a_x b_v-b_x a_v)+(a_y b_v-b_y a_v)]$ ...(87)

As is shown above, this model satisfies the general vector equality demonstrated in Gibbs (1884) for Dot triple product: $\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})=\mathbf{b}\cdot(\mathbf{c}\times\mathbf{a})=\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$:

Or: $\mathbf{d} = \begin{vmatrix} c_x & c_y & c_v & c_w \\ a_x & a_y & a_v & a_w \\ b_x & b_y & b_v & b_w \end{vmatrix} = \begin{vmatrix} b_x & b_y & b_v & b_w \\ c_x & c_y & c_v & c_w \\ a_x & a_y & a_v & a_w \end{vmatrix} = \begin{vmatrix} a_x & a_y & a_v & a_w \\ b_x & b_y & b_v & b_w \\ c_x & c_y & c_v & c_w \end{vmatrix}$

$= c_x[(a_y b_v - b_y a_v) + (a_y b_w - b_y a_w) + (a_v b_w - b_v a_w)] + c_y[-(a_x b_w - b_x a_w) - (a_x b_v - b_x a_v) + (a_v b_w - b_v a_w)] + c_v[(a_x b_y - b_x a_y) - (a_x b_w - b_x a_w) - (a_y b_w - b_y a_w)] + c_w[(a_x b_y - b_x a_y) + (a_x b_v - b_x a_v) + (a_y b_v - b_y a_v)]$ ...(88)

Namely, in a space of four dimensions exists a lot of figures like the rectilinear line represented in one, two, three and four dimensions; two-dimensional curve lines, three-dimensional curve lines and also four-dimensional curve lines, two dimensional surfaces, three-dimensional surfaces and also four-dimensional surfaces, three-dimensional parallelepipeds and also four-dimensional parallelepipeds and many other figures familiar or unfamiliar to us. In the case above we were analyzing a scalar triple product which gives us, as said, the three-dimensional volume of a parallelepiped of sides **a**, **b**, and **c** inside a four-dimensional space.

8.3. Three-Dimensional Dot Product of Three Complex Vectors

$\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b}) = \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) = \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$

The same general procedure for Dot triple product applied to pure vectors is used on three “3D” (6D)-complex vectors **a**, **b** and **c**, for: $\mathbf{a} = (a_x + ja_X)\mathbf{i} + (a_y + ja_Y)\mathbf{j} + (a_z + ja_Z)\mathbf{k}$, $\mathbf{b} = (b_x + jb_X)\mathbf{i} + (b_y + jb_Y)\mathbf{j} + (b_z + jb_Z)\mathbf{k}$, $\mathbf{c} = (c_x + jc_X)\mathbf{i} + (c_y + jc_Y)\mathbf{j} + (c_z + jc_Z)\mathbf{k}$. By applying the following 4x6 direct Dot triple-product matrix of real and imaginary vectors the result of $C_{5,2}$=10 matrices (2x2) per each factor c_m (for $m = x, y, z, X, Y, Z$) is obtained directly:

$$\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b}) = \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) = \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \begin{vmatrix} c_x & c_y & c_z & c_X & c_Y & c_Z \\ a_x & a_y & a_z & a_X & a_Y & a_Z \\ b_x & b_y & b_z & b_X & b_Y & b_Z \end{vmatrix} = \begin{vmatrix} b_x & b_y & b_z & b_X & b_Y & b_Z \\ c_x & c_y & c_z & c_X & c_Y & c_Z \\ a_x & a_y & a_z & a_X & a_Y & a_Z \end{vmatrix} = \begin{vmatrix} a_x & a_y & a_z & a_X & a_Y & a_Z \\ b_x & b_y & b_z & b_X & b_Y & b_Z \\ c_x & c_y & c_z & c_X & c_Y & c_Z \end{vmatrix}$$

$= c_x[(a_y b_z - b_y a_z) + (a_y b_X - b_y a_X) + (a_y b_Y - b_y a_Y) + (a_y b_Z - b_y a_Z) + (a_z b_X - b_z a_X) + (a_z b_Y - b_z a_Y) + (a_z b_Z - b_z a_Z) + (a_X b_Y - b_X a_Y) + (a_X b_Z - b_X a_Z) + (a_Y b_Z - b_Y a_Z)] + c_y[-(a_x b_z - b_x a_z) - (a_x b_X - b_x a_X) - (a_x b_Y - b_x a_Y) - (a_x b_Z - b_x a_Z) + (a_z b_X - b_z a_X) + (a_z b_Y - b_z a_Y) + (a_z b_Z - b_z a_Z) + (a_X b_Y - b_X a_Y) + (a_X b_Z - b_X a_Z) + (a_Y b_Z - b_Y a_Z)] + c_z[(a_x b_y - b_x a_y) - (a_x b_X - b_x a_Z) - (a_x b_Y - b_x a_Y) - (a_x b_Z - b_x a_Z) - (a_y b_X - b_y a_X) - (a_y b_Y - b_y a_Y) - (a_y b_Z - b_y a_Z) + (a_X b_Y - b_X a_Y) + (a_X b_Z - b_X a_Z) + (a_Y b_Z - b_Y a_Z)] + c_X[(a_x b_y - b_x a_y) + (a_x b_z - b_x a_z) + (a_y b_z - b_y a_z) - (a_x b_Y - b_x a_Y) - (a_y b_Y - b_y a_Y) - (a_z b_Y - b_z a_Y) - (a_x b_Z - b_x a_Z) - (a_y b_z - b_y a_z) - (a_z b_Y - b_z a_Z) + (a_Y b_Z - b_Y a_Z)] + c_Y[(a_x b_y - b_x a_y) + (a_x b_z - b_x a_z) + (a_x b_X - b_x a_X) + (a_y b_z - b_y a_z) + (a_y b_X - b_y a_X) + (a_z b_X - b_z a_X) - (a_x b_Z - b_x a_Z) - (a_y b_Z - b_y a_Z) - (a_z b_Z - b_z a_Z) - (a_X b_Z - b_X a_Z)] + c_Z[(a_x b_y - b_x a_y) + (a_x b_z - b_x a_z) + (a_x b_X - b_x a_X) + (a_x b_Y - b_x a_Y) + (a_y b_z - b_y a_z) + (a_y b_X - b_y a_X) + (a_y b_Y - b_y a_Y) + (a_z b_X - b_z a_X) + (a_z b_Y - b_z a_Y) + (a_X b_Y - b_X a_Y)]$ ...(89)

9. Complex Vectors: Summary (and with the Imaginary Part Null for Pure Vectors)

From results, and definitions in sections above, can be derived the following properties for complex vectors, additional to those known of pure vectors.

A) $\mathbf{r} = (a + jb)\mathbf{i} + (c + jd)\mathbf{j} + ... + (s + jt)\mathbf{n} = (a\mathbf{i} + c\mathbf{j} + ... + s\mathbf{n}) + [b(j\mathbf{i}) + d(j\mathbf{j}) + ... + t(j\mathbf{n})]$. A basis vector **v** multiplied by $j = \sqrt{-1}$ creates a new basis vector $\hat{\boldsymbol{v}} = j\mathbf{v}$, or an imaginary vector $\hat{\boldsymbol{v}}$, perpendicular to **v**.

B) $\mathbf{r} = \boldsymbol{\rho} + \hat{\boldsymbol{\gamma}}$; $\boldsymbol{\rho} = a\mathbf{i} + c\mathbf{j} + ... + s\mathbf{n}$ and $\hat{\boldsymbol{\gamma}} = b(j\mathbf{i}) + d(j\mathbf{j}) + ... + t(j\mathbf{n}) = b\,\hat{\boldsymbol{i}} + d\,\hat{\boldsymbol{j}} + ... + t\,\hat{\boldsymbol{n}}$

C) $\mathbf{r} = r\mathbf{u}_r = \rho\mathbf{u} + \gamma\hat{\boldsymbol{u}}$, where **u** and $\hat{\boldsymbol{u}}$ are perpendicular unit vectors.

For $r = \sqrt{\rho^2 + \gamma^2}$; $\rho = \sqrt{a^2 + c^2 + \ldots + s^2}$ and $\gamma = \sqrt{b^2 + d^2 + \ldots + t^2}$.

So, $r = \sqrt{a^2 + c^2 + \ldots + s^2 + b^2 + d^2 + \ldots + t^2}$.

D) $\mathbf{r} = \boldsymbol{\rho} + \hat{\boldsymbol{\gamma}} = \rho\mathbf{u} + \gamma\hat{\boldsymbol{u}} = r\mathbf{u}_r \rightarrow \mathbf{r} = r(\cos\theta\mathbf{u} + \sin\theta\hat{\boldsymbol{u}}) = re^{j\theta}\mathbf{u}_r$, where **r**, inside a space of $2N$ dimensions, is located in the plane formed by the perpendicular vectors $\boldsymbol{\rho}$ and $\hat{\boldsymbol{\gamma}}$; and θ is the angle between **r** and $\boldsymbol{\rho}$. In this way, we can write that $\rho = r\cos\theta$ and $\gamma = r\sin\theta$.

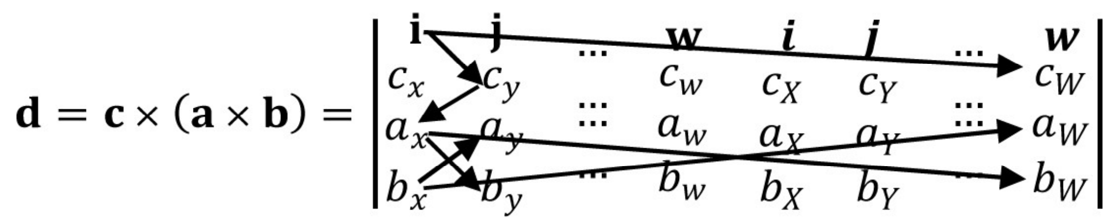


Figure 6: Graph Showing the Way to Proceed with any 4x2N Matrix of Complex Vectors

E) Dot product for basis vectors where $\mathbf{p} \neq \mathbf{q}$: $\mathbf{p}.\mathbf{q} = 0$; $\mathbf{p}.\mathbf{p} = \mathbf{q}.\mathbf{q} = 1$ for $\mathbf{p}$ and $\mathbf{q} = \mathbf{i}, \mathbf{j}, \mathbf{n}, \ldots, \hat{\boldsymbol{i}}, \hat{\boldsymbol{j}}, \hat{\boldsymbol{n}}$.

F) $\boldsymbol{v} = z(k(\boldsymbol{r})) = z(k(re^{j\theta}\boldsymbol{u}_r)) = kz(re^{j\theta}\boldsymbol{u}_r)$, where k is a scalar and $z = |z|e^{j\varphi}$ a complex number. In addition to multiplying by k, a rotation φ of the vector $\mathbf{r}$ is produced by z: i.e.: $\mathbf{v} = |z|e^{j\varphi}\left(k\left(re^{j\theta}\mathbf{u}_r\right)\right) = k|z|\left(re^{j(\theta+\varphi)}u_r\right) = k|z|\left(r\left(\cos(\theta+\varphi)\boldsymbol{u} + \sin(\theta+\varphi)\hat{\boldsymbol{u}}\right)\right)$.

G) Multi-Dimensional Dot product of TWO Complex Vectors:

$$\mathbf{r_1} \cdot \mathbf{r_2} = \sum_1^N \left(x_{1i}x_{2i} + y_{1i}y_{2i}\right)$$

H) Multi-Dimensional Cross product of TWO Complex Vectors:

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} & \cdots & \mathbf{n} & \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} & \cdots & \hat{\boldsymbol{n}} \\ x_{11} & x_{12} & x_{13} & \cdots & x_{1N} & y_{11} & y_{12} & y_{13} & \cdots & y_{1N} \\ x_{21} & x_{22} & x_{23} & \cdots & x_{2N} & y_{21} & y_{22} & y_{23} & \cdots & y_{2N} \end{vmatrix} = \text{sum of all the 2x2 matrices:}$$

$$\begin{vmatrix} \mathbf{u}_{]ab[} & \mathbf{a} & \mathbf{b} \\ 0 & x_{1a} & x_{2a} \\ 0 & x_{2a} & x_{2b} \end{vmatrix} \text{ for } \begin{Bmatrix} \mathbf{a} = \mathbf{i}\ldots\mathbf{n}, \textit{ and } \mathbf{b} = \mathbf{j}\ldots\mathbf{n} \\ a = 1\ldots N, \textit{ and } b = 2\ldots N \end{Bmatrix}, \begin{vmatrix} \mathbf{u}_{]a\hat{d}[} & \mathbf{a} & \hat{\boldsymbol{d}} \\ 0 & x_{1a} & y_{1\hat{d}} \\ 0 & x_{2a} & y_{2\hat{d}} \end{vmatrix} \text{ for } \begin{Bmatrix} \mathbf{a} = \mathbf{i}\ldots\mathbf{n}, \textit{ and } \hat{\boldsymbol{d}} = \hat{\boldsymbol{i}}\ldots\hat{\boldsymbol{n}} \\ a = 1\ldots N, \textit{ and } \hat{d} = 1\ldots N \end{Bmatrix}$$

$$\text{or } \begin{vmatrix} \mathbf{u}_{]\hat{c}\hat{d}[} & \hat{\boldsymbol{c}} & \hat{\boldsymbol{d}} \\ 0 & y_{1\hat{c}} & y_{1\hat{d}} \\ 0 & y_{2\hat{c}} & y_{2\hat{d}} \end{vmatrix} \text{ for } \begin{Bmatrix} \hat{\boldsymbol{c}} = \mathbf{i}\ldots\mathbf{n}, \textit{ and } \hat{\boldsymbol{d}} = \hat{\boldsymbol{j}}\ldots\hat{\boldsymbol{n}} \\ \hat{c} = 1\ldots N, \textit{ and } \hat{d} = 1\ldots N \end{Bmatrix}$$

I) 2N-Dimensional Dot product of THREE Complex Vectors satisfying:

$$\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$$

$$\mathbf{d} = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) = \begin{vmatrix} c_x & c_y & \ldots & c_w & c_X & c_Y & \ldots & c_W \\ a_x & a_y & \ldots & a_w & a_X & a_Y & \ldots & a_W \\ b_x & b_y & \ldots & b_w & b_X & b_Y & \ldots & b_W \end{vmatrix}$$

. See sub-sections 8.2 and 8.3.

J) 2N-Dimensional Cross product of THREE Complex Vectors satisfying:

$$\mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = (\mathbf{c} \cdot \mathbf{b})\mathbf{a} - (\mathbf{c} \cdot \mathbf{a})\mathbf{b}$$

Doing as indicated in Figure 6, changing the initial column to the last one and repeating we obtain the resultant sum of matrices.

(See sub-sections 7.2 and 7.4.)

10. Conclusion

New expressions for Multidimensional Cross and Dot products of Complex Vectors have been achieved, by using known general relations of pure vectors Gibbs (1884) and showing how the expression of complex roots on each axis is obtained as a natural solution to the strategic problem of cutting a parabola with its directrix (they never cut each other). With these foundations we defined and achieved the calculation the Multidimensional Dot product of two Complex

Vectors and , giving as result a real number; and also, of its Multidimensional Cross Product, yielding another complex vector, perpendicular to $\mathbf{r}_1$ and $\mathbf{r}_2$. The used matrices of calculation give consistent results for both the Dot and Cross Product of two pure or complex vectors, considered them as pure or complex lines ($\mathbf{r}_1 . \mathbf{r}_2 = r_1 r_2 \cos\theta$, $\mathbf{r}_1 \text{ x } \mathbf{r}_2 = r_1 r_2 \sin\theta\, \mathbf{n}$). Based on the general definitions indicated above and those found in Gibbs (1884) it was also possible to get the Multi-Dimensional Dot and Cross complex triple products, **c** · (**a** x **b**) and **c** x (**a** x **b**). Similar analyses can be developed in detail (as in this work) and checked for other N-Dimensional Dot and Cross products with general relations provided in Gibbs (1884), such as:

a) $[\alpha \times \beta] . [\gamma \times \delta] = (\alpha . \gamma)(\beta . \delta) - (\alpha . \delta)(\beta . \gamma)$

b) $[\alpha \times \beta] \times [\gamma \times \delta] = (\alpha . \gamma \times \delta)\beta - (\beta . \gamma \times \delta)\alpha = (\alpha . \beta \times \delta)\gamma - (\alpha . \beta \times \gamma)\delta$

c) $\alpha \times [\beta \times [\gamma \times \delta]] = (\alpha . \gamma \times \delta)\beta - (\alpha . \beta)\gamma \times \delta = (\beta . \delta)\alpha \times \gamma - (\beta . \gamma)\alpha \times \delta$

d) $[\alpha \times \beta] . [\gamma \times \delta] \times [\varepsilon \times \xi] = (\alpha . \beta \times \delta)(\gamma . \varepsilon \times \xi) - (\alpha . \beta \times \gamma)(\delta . \varepsilon \times \xi)$

$= (\alpha . \beta \times \varepsilon)(\xi . \gamma \times \delta) - (\alpha . \beta \times \xi)(\varepsilon . \gamma \times \delta) = (\gamma . \delta \times \alpha)(\beta . \varepsilon \times \xi) - (\gamma . \delta \times \beta)(\alpha . \varepsilon \times \xi)$

e) $[\alpha \times \beta] . [\beta \times \gamma] \times [\gamma \times \alpha] = (\alpha . \beta \times \gamma)^2$.

Finally, this work shows that the definition of "the conjugate" of a complex number is a forced (false) concept that leads to unsolvable contradictions, not only with the definition of the modulus of a complex vector, but may be one of the causes that have so far prevented obtaining an adequate definition of the complex vector itself (such as the one achieved in Section 2.2).

Acknowledgment

I thank Prof. José G. Quintero M., for his helpful suggestions and recommendations, Prof. J. Conesa M., for his mathematical support and time dedicated to the review, Prof. Andrés Echezuría Q., for his help with the Sagemath routines, Prof. E. Tescari James for his proofreading work (all of them retirees, like me), and finally Nora Franco, Noreli Franco and Nadia Franco for their invaluable help in writing, and their encouragement and solidarity in and during the conduct of this research. In addition, this work has been carried out thanks to the useful Sagemath software and to the resources provided by the Department of Mathematics of the Universidad Autónoma de Madrid, for the on-line calculations. The use of DeepL Translator was also a great help to check and correct the English.

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Cite this article as: Jorge A. Franco (2023). Multidimensional Complex Vectors. *International Journal of Pure and Applied Mathematics Research*, 3(2), 1-19. doi: 10.51483/IJPAMR.3.2.2023.1-19.