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Fermat's Last Theorem Proved in Hilbert Arithmetic I: From the Proof by Induction to the Viewpoint of Hilbert Arithmetic

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Abstract

In a previous paper (https://dx.doi.org/10.2139/ssrn.3648127), an elementary and thoroughly arithmetical proof of Fermat's last theorem by induction has been demonstrated if the case for "n = 3" is granted as proved only arithmetically (which is a fact a long time ago), furthermore in a way accessible to Fermat himself though without being absolutely and precisely correct. The present paper elucidates the contemporary mathematical background, from which an inductive proof of FLT can be inferred since its proof for the case for "n = 3" has been known for a long time. It needs "Hilbert mathematics", which is inherently complete unlike the usual "Gödel mathematics", and based on "Hilbert arithmetic" to generalize Peano arithmetic in a way to unify it with the qubit Hilbert space of quantum information. An "epoché to infinity" (similar to Husserl's "epoché to reality") is necessary to map Hilbert arithmetic into Peano arithmetic in order to be relevant to Fermat's age. Furthermore, the two linked semigroups originating from addition and multiplication and from the Peano axioms in the final analysis can be postulated algebraically as independent of each other in a "Hamilton" modification of arithmetic supposedly equivalent to Peano arithmetic. The inductive proof of FLT can be deduced absolutely precisely in that Hamilton arithmetic and the pransfered as a corollary in the standard Peano arithmetic furthermore in a way accessible in Fermat's epoch and thus, to himself in principle. A future, secon d part of the paper is outlined, getting directed to an eventual proof of the case "n = 3" based on the qubit Hilbert space and the Kochen-Specker theorem inferable from it.

Keywords: *Fermat's last theorem, Hilbert arithmetic, Kochen and Specker's theorem, Peano arithmetic, Quantum information, Qubit Hilbert space*

1. The Fruitful Tension of Idempotency and Hierarchy, or about its Embodiment in the Separable Complex Hilbert Space of Quantum Mechanics as Unitarity

Fermat's Last Theorem (or FLT further for briefly) is often considered to be a mathematical "curio" and the centuries-old troubles about its correct proof to be inexplicable and even ridiculous. Even Wiles's proof (1995) elucidates its meaning as rather an artifact being a direct corollary from the modularity theorem which is properly the very important

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one for mathematics being able to link the discrete mathematics of modular forms and the continuous (even smooth, i.e., differentiable) mathematics of elliptic curves. In fact, that implicit understanding about the meaning of FLT follows its realization as a specific, proper arithmetic statement without any essential methodological connotations (unlike the parent theorem of modularity) and thus without any influence to philosophy or, at least, to philosophy of mathematics[1].

The advocated approach is quite different: it interprets FLT as relevant to the foundations of mathematics as implied by the relation of arithmetic and set theory and thus obeying the Gödel dichotomy about it: either incompleteness or contradiction. In other words, FLT is suggested to be a Gödel insoluble statement if the relevant framework is granted to be that of arithmetic and set theory as first-order logics to propositional logic, i.e., the usual and traditional basis of mathematics, furthermore able to supply models for any any theory belonging to any branch of it.

Even more, the same structure of FLT (by the way, Fermat's equation is especially symmetric, unlike almost all Diophantus equations, to the class of which it belongs) can be observed as an embodiment of the relation of arithmetic and set theory, in turn and paradoxically hinting at its proof as a hidden key or a secret code contained in itself. In other words, a proper philosophical approach relying on its "mathematical semiotics" or "hermeneutics" is suggested, but not as a basis for loose, "humanitarian" interpretations contradicting the fundamental principles of logic and mathematics, and even science at all; on the contrary, an absolutely rigorous proof consistent to the enumerated foundations of mathematics is meant:

It is grounded in Hilbert arithmetic discussed in other papers (e.g., Penchev, 2020; 2021), which can be furthermore used as a "Wittgenstein ladder"[2] for an only arithmetical proof of FLT; at that relatively elementary as an ultimate result and thus accomplishable by Fermat himself and verifying the option of his own "lost proof". In the final analysis, the tools, to which one can restrict the necessary formal and logical means after removing the "Wittgenstein ladder" of Hilbert arithmetic, turn out to be only two: (1) modified *modus tollens*, notated further and briefly as "MMT"; and (2) mathematical induction in the specific form of Fermat's descent (or "MFD" further). Obviously, both might be used by Fermat himself.

Anyway, that approach though accomplishable in Fermat's age in virtue of its "inexperience" has become more and more impossible after Descartes's dualism gradually limiting the site of mathematics in the organization of knowledge and cognition (or Michel Foócault's "episteme" suggested in his book in 1966), to "mind", and after set theory, explicitly formulated and enumerated among the foundations of mathematics.

Thus, a "bracket" touching Wiles's proof can be opened: since it infers FLT from the modularity theorem, set theory is necessarily involved, and consequently, the utilized branches of mathematics are to go out from the standard mathematics (if it is restricted as usual to the framework underlain by arithmetic and set theory). The noticed contradiction of Wiles's proof and the sketched here approach needs an explicit proof discussed in detail in Section IV "A formal proof of the insolubility of FLT in Gödel mathematics (arithmetic and set theory)" and elucidating that FLT is a Gödel insoluble statement by the mediation of Yablo's paradox[3], which the eventual proof of FLT in the framework of both arithmetic and set theory obeys necessarily.

Simultaneously, one can interpret the "inexperience" of Fermat's age also as a pre-bifurcational state in history of mathematics forked in our real branch inspired by Descartes's dualism and culminated into the establishment of set theory, therefore ultimately closed any option for the alleged original and only arithmetical proof of Fermat. Thus, a counterfactual branch is not less suggestable, in which Fermat's proof would not be "lost", and would march triumphantly,

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[1] An "exception that proves the rule" is the book by Schwarzer (2020) who had written that "there must be a connection between Fermat's last theorem and quantum theory" (p. 101) and this is its basic assumption.

[2] The expression "Wittgenstein ladder" is frequently used and investigated in the context of Wittgenstein's works (e.g., Reid, 1998 and Ware, 2015). Furthermore, it is also utilized sometimes out of that context as a metaphor, but anyway in the framework of philosophy (e.g., Perloff, 1996) However, the term in the present paper serves rather to designate a philosophical and mathematical method related to Husserl's "epoché". So, its philosophical meaning can be inferred from that comparison, and as a proper mathematical approach, it means a solution of a problem to be researched in a structure more general (i.e., without certain concepts or axioms) than that meant in default. In the case of FLT, it suggests only arithmetic (rather than arithmetic and set theory) to be used for its proof.

[3] Yablo's paradox suggests a wide mathematical and philosophical context relevant to the present paper and elaborated by Yablo himself along with many other researchers. He discusses proper philosophical ideas in many papers (e.g., Yablo, 1982; 1985; 1987; 1992a; 1992b; 1993a; 1993b; 1993c; 1997a; 1997b; 2000a; 2000b; 2002; 2003; 2005a; 2005b; 2016). So one can speak of Yablo's philosophy relevant to Yablo's paradox. Though the former alone can be an instructive and fruitful subject of study, it is out of the scope of the present investigation anyway. Nonetheless, his philosophical ideas are meant implicitly, but without expressive references since the relations are mediated and often sophisticated.

being so familiar to any student there as the Pythagorean theorem or as the concept of “set” following our own branch in the history of mathematics. On the contrary, one can admit that “set” would be just so “lost” in that counterfactual branch as Fermat’s original proof in ours.

One can notice the horizon marked in the present article as a rigorous “counterfactual history of mathematics” based în axioms whether inexistent yet or being negations of existent ones. Thus, the paper can be enumerated also among the first representatives of that yet absent “genre” of “counterfactual mathematics” closely linked to its philosophy and foundations.

If one “gazes” at the Gödel dichotomy about the relation of arithmetic and set theory (e.g., as in: Penchev, 2016; 2020), one can localize its reason only in the direct contradiction of the axiom of induction (valid in arithmetic) and the axiom of infinity (postulated in set theory) in relation of all natural numbers: indeed, the axiom of induction implies for all natural numbers to be finite, but the axiom of infinity states that the *set* of all natural numbers is infinite[4]. Thus, the axiom of infinity introduces a special quality to any “naive” collection homogenous in a sense, namely to be a “set” opposing it to the quality of its elements.

One can object immediately that FLT is an only arithmetical statement so that the influence of set theory on it is rather doubtful. However, FLT is a statement referring to *all* exponents (natural numbers) greater than 2, therefore implying the problem of how those “all exponents” are to be interpreted: whether as natural numbers or as the set of them. According to the proper arithmetical formulation of FLT, they are to be “natural numbers” rather than as their “set”, however this implies an indirect problem consisting in the following: the proof about all natural numbers greater than two needs the set of all natural numbers than two, though that set can be removed in the ultimate result just as a “Wittgenstein ladder”:

The observation at issue is an exemplification of the statement justified in another paper (Penchev, 2021) and stating figuratively that “Achilles will never overtake the Tortoise” in an only “arithmetic universe” once it is interpreted in logical terms as Carroll (1895) did. Thus, the proof of FLT, though formulated only arithmetically, needs set theory just like Achilles if he “wishes” to overtake the Tortoise (which our usual experience reckons to be obvious). In other words, the statement about all natural numbers greater than two need be “overtaken” (or as if “seen outside”) to be proved, and this only set theory, or at least the concept of set, is able to provide or guarantee. Thus, set theory though hidden behind propositional logic turns out to be necessary for proving the (otherwise absolutely) arithmetical statement of FLT:

Nonetheless, the mediation of set theory is simultaneously impossible just by virtue of the Gödel dichotomy. So, one can formulate explicitly a “paradox of the proof of FLT” summarizing both opposite tenets above and consisting in the following:

(1) On the one hand, the proof of FLT needs the set of all natural numbers greater than two and thus set theory, though FLT is an only arithmetic statement referring to all natural numbers rather than to the set of them.

(2) However on the other hand, FLT is a Gödel insoluble statement in the framework of both arithmetic and set theory, so adding set theory FLT cannot be proved again.

Formally, those, i.e., both (1) and (2) above, seem not to be a real logical paradox, but rather a statement contradicting common sense’s belief or at least, the mathematical common sense’s belief: namely, though Gödel’s proof of the incompleteness theorems is constructive (the opposite thesis is advocated in: (Penchev, 2010; 2020), there do not exist any examples of Gödel insoluble statements in the real history of mathematics, so that any mathematical statement met by humankind is either true or false, and thus always soluble. Consequently, FLT “should be soluble”, too.

The present paper supports the same intuitive conviction of many mathematicians, however following a proper mathematical, formal and logical way, in which the Gödel “first incompleteness theorem”[5] (“Satz VI” in his original paper in 1931) is justified to be an independent axiom[6], incorrectly interpreted as a theorem in virtue of pre-predicative and

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[4] “The Status of induction” (Detlefsen, 1986) is the central problem in the relation of Hilbert’s program and the discussed approach to FLT. The same author (Detlefsen, 1990) clarifies why Gödel’s first incompleteness theorem (i.e., “Satz VI” in the original paper) does not refute Hilbert’s program in a way relative to ideas here.

[5] So-called “second incompleteness theorem” is granted to be inferable from the first one. FLT needs quantifiers to be formulated: so it is out of “elementary arithmetic” without quantifiers. Nonetheless the second incompleteness theorem seems to be reformulatble in that “elementary arithmetic” (Pozsgay, 1968). However, if the second incompleteness theorem is generalized to general recursive arithmetic (as in Ryan, 1978), FLT can be also generalized to it.

non-articulated pråjudice out of the proper scope of mathematics and delivered by the fundamental organization of knowledge and cognition in Modernity, i.e., Foucault (1966)'s "episteme", about the subordinate position of mathematics in it.

On the contrary, if that "theorem" is not granted (which is quite admissible since it is an independent axiom) just only that subordinate position of mathematics is rejected and if that is the case, the world is a true part from mathematics (rather than vice versa), and insoluble statements really do not exist just according to the intuitive conviction of almost all mathematicians. That mathematics renouncing its subordination (and formally, the "Gödel incompleteness axiom" replacing it by its negation) is called *Hilbert mathematics*[7].

Then, and unlike the alternative *Gödel mathematics*, the sketched above paradox about the proof of FLT is a real paradox since insoluble statements cannot exist in Hilbert mathematics. Consequently, it needs a relevant explanation of itself in Hilbert mathematics to be conserved as consistent, to which, particularly, the present paper is devoted.

The meant way out, quite concisely since it is described many times in details, including in the present paper, and if it is to be related just to FLT, consists in its proof in Hilbert arithmetic relevant to Hilbert mathematics unlike Gödel mathematics, eventually demonstrating that the proof can be mapped unambiguously (i.e., mathematically homomorphically) into Peano arithmetic though in a rather sophisticated approach, nonetheless supplying FLT by a proof in Peano arithmetic and even accessible to Fermat himself therefore supporting the option of his original, but "lost" proof, since that complicated approach at issue can be removed from the ultimate syllogism of the proof like a "Wittgenstein ladder", or speaking more precisely, by virtue of Gentzen's (1935) "cut-elimination" . Particularly, the idea of an only arithmetic proof of FLT is consistent even to Gödel mathematics, since his incompleteness theorems are valid only in the framework of both arithmetic and set theory, but not only within arithmetic alone.

However, the problem of how the only arithmetical proof of FLT avoids the paradox of its proof, and more especially, the necessity of the *set* of all natural numbers greater than two, is to be expressively shown. One should first make clear where the set of all natural numbers (or any infinite subset of it) is really necessary in an only arithmetical proof (such as the intended one of FLT), on the one hand, and how a statement relating to all natural numbers (excluding eventually some of them e.g., as in FLT) can be proved at all in arithmetic, on the other hand.

In fact, the proper arithmetic necessity of the set of all natural numbers (or a relevant subset) is rather ambiguous for the cut-elimination rule: there exists always a natural number (thus necessarily being a finite number) of implications, to which any proof involving an infinite subset of the set of all natural numbers can be reduced, however this is a pure proof of existence alone or at least, a general method of how that finite number of implications to be revealed explicitly is not known[8]. So, that infinite set of natural numbers in an arithmetical proof means only the existence of the proof, but not any finite constructive sequence to be really accomplished.

That kind of use of an infinite set can be visualized by Carroll's logical metaphor of "What the Tortoise said to Achilles": the infinite set guarantees that Achilles will overtake or has overtaken the Tortoise (a statement, which corresponds to our experience absolutely), however bracketing the exact moment of time when he is overtaking the Tortoise "constructively" since that moment generates insurmountable contradictions (just the subject of Zeno's paradox). Quite analogically or even isomorphically, any implication (properly meant by Carroll's paradox) can be accomplished by the mediation of the set of an infinite series of implications, though the exact moment of its accomplishment meets the same kind of insoluble contradictions as those in Zeno's paradox.

To that set eventually involved in an arithmetical proof referring to an infinite set of natural numbers, two fundamental, proper arithmetical tools correspond: mathematical induction according to the axiom of induction and *reductio ad absurdum* after the assumption that the set at issue is finite (e.g., as in the proof that the prime numbers are an infinite

[6] For example, Akiyoshi (2009) demonstrates the viewpoint, from which one "cannot claim that Gödel's second incompleteness theorem defeats Hilbert's program. Moreover we clarify what is essentially needed for such an argument to succeed".

[7] A paper (Barrio, 2010) can be considered as a counterargument to "Hilbert mathematics" as far it "is to show that it's not a good idea to have a theory of truth that is consistent but ω-inconsistent", and Hilbert mathematics can be interpreted to suggest that kind of truth, and as to FLT by means of Yablo's paradox also utilized in the paper at issue. However, if one accepts a Pythagorean ontology of the world (as here) that theory of truth seems to be admissible.

[8] Anyway, one may mention certain modifications of Gödel's "T system" (e.g., Alves *et al.*, 2010) presumably able to overcome that restriction. Furthermore, the approach of Kanskos (2010) to the consistency of Heyting arithmetic to be proved by "an interpretation of the expressions of the vectors as ordinals" is rather similar to the interpretation of transfinite numbers as vectors in a relevant kind of Hilbert space as in Hilbert arithmetic.

set). One should pay attention that both are postulated (the former as an arithmetic axiom, and the latter as a corollary from *modus tollens*) and both mean an unambiguous transition back over the contradictory limit between finiteness and infinity, and overcome by both "physical Achilles" in Zeno's paradox and "logical Achilles" in Carroll's paradox in the "straight" direction without problems as to any real experience, and finally, both necessary for the eventual only arithmetic proof of FLT (though not *reductio ad absurdum*, but its parent *modus tollens*).

Speaking rather loosely or figuratively, one might say, all centuries old troubles about FLT are due to the necessity to be used both means in a single proof as if passing back over the limit between finiteness and infinity *two times*: the one time by *modus tollens* and one more time by the axiom of induction. The reason for that extraordinary *double* necessity consists, in the final analysis, just in the fact that FLT is a Gödel insoluble statement. Speaking again rather figuratively, the first leap over the finiteness/infinity limit is necessary for it to be transformed from an insoluble proposition into a usual one, i.e., either false or true, but never both; and the second one, for a relevant natural (i.e., finite) natural number to be chosen unambiguously to the intermediate soluble propositions (i.e., being a hypothetical soluble counterpart of FLT, which is insoluble by itself).

The same double transition over the finiteness/infinity limit can be very easily visualized by the consideration in terms of Hilbert arithmetic: the first transition transfers from a finite number, but belonging to the dual Peano arithmetic into the shared "arithmetic infinity"; then, the second transition does the same, but as if in an opposite direction, namely, from the shared "arithmetic infinity" in the initial Peano arithmetic.

However, even that very sophisticated construction is yet insufficient for the ultimate, only arithmetical proof of FLT: it needs the axiom of choice, furthermore utilizing it by the vanishing way of a "Wittgenstein ladder"[9], for the exchange the order of the two transitions over the finiteness/infinity limit and directed oppositely to each order: after the exchange of the order, both transitions share the same direction or speaking conventionally: from infinity to finiteness. Once all those very complicated preparations have been ultimately accomplished, the "Wittgenstein laffer" of Hilbert arithmetic can be absolutely removed and thoroughly hidden, so that an only arithmetical proof of FLT can occur as Fermat's "lost proof".

A still more abstract and philosophical basis of the option for FLT to be proved only arithmetically can be extracted further. It refers only to the metarelation (which can be alleged or called "formal completeness") of idempotency and hierarchy (i.e., well-ordering, but meant simultaneously more philosophically[10]). From the viewpoint advocated in the present paper, the problem of a pure arithmetic proof of FLT is elementary within a kind of "complete arithmetic", which Hilbert arithmetic claims to be. In others words, all the "vanity" about FLT is due to the inherent incompleteness of Gödel mathematics, which is not available in Hilbert mathematics: at that, in a natural way.

As to Fermat's age, the bifurcation of Gödel mathematics versus Hilbert mathematics had not yet taken place and one can speak about a kind of "Fermat arithmetic", in which Peano axioms were not available as well as set theory or even only the concept of "infinite set": natural number could be yet "infinite" in a naive and non-articulated sense and not opposed the usual finite natural numbers realized in our epoch.

Meaning that "Fermat arithmetic", one can depict and then map formally and rigorously the proof from Hilbert arithmetic, which in it is possible in a way not so unimaginably difficult as following Wiles (1995), but rather elementarily, into Fermat arithmetic at issue in a very sophisticated way, but anyway removable in the ultimate result (which is the alleged "lost proof" of Fermat) just as a "Wittgenstein ladder", and which is the subject of the present study.

Mentioning Wiles's proof, accomplished ostensibly thoroughly in the framework of Gödel mathematics, one can question what his value in Hilbert mathematics is. In fact, it cannot be really made in Gödel mathematics as this is an insoluble statement in it: and as it will be proved rigorously in *Section IV*. So, the real sense of Wiles's proof is to introduce implicitly or to "sneak in" quietly through the backdoor the framework of Hilbert mathematics by a relevant equivalent of the complete Hilbert arithmetic (or eventually weaker than it, but sufficient for the proof of FLT) in a way so exceptionally complicated that only a handful of mathematicians can indeed realize what is done therefore avoiding

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9 That use of the axiom of choice as a "Wittgenstein ladder" can be related to the interpretation of Hilbert's epsilon operator as its implicite action within arithmetic (without set theory), for example as by Wirth (2008).

10 Particularly, the distinction investigated by Freyd (1985) is not relevant. "It is one thing to say that the axiom of choice implies that everything may be well-ordered. It is quite another thing to say that if a set has a choice function then it has a well-ordering." Furthermore, the distinctions in computer science (e.g., Hornby, 2007) should not be meant as well. Yagil (1999) investigates the link of hierarchy to complexity, and Zheng, Weihrauch (2001): how real numbers can be represented by arithmetical hierarchy (meaning that all real numbers are well-ordered sets of natural numbers, *infinite* in general).

the cries of the “Boeotians” (of whom Gauss feared and therefore did not publish his work on non-Euclidean geometry) if anyone dare encroach the general organization of cognition in Modernity, and more especially, where mathematics is to be situated, however absolutely necessary if that one (like Wiles) wishes to prove FLT.

One can figuratively represent Wiles’s approach on the background of the “parable of the Nude King”. Only a lad and commoner can “scream out”: “The king is nude!”. However, Sir Andrwew Wiles is an exceptionally respectable scientist, to say, as if rather belonging to the retinue of the king than a lad and commoner. Of course, he sees that the king is nude and wants to proclaim that loudly, but he can do this only in a way relevant to his social status and using a very sophisticated language understandable only for the other “tailors” and thus not calling for a revolution of cognition (a behavior absolutely inadmissible as to a respectable scientist).

One can verify that Gauss, being another respectable scientist, chose to remain silent, and only Lobachewski, a scientist in the Kazan university in Russia, i.e., almost out of Europe or at its boundary to Asia might have the courage to do so at all. Poincaré, another respectable scientist a century later anticipated the approach of Wiles just speaking to the other “tailors” but without any revolutionary ontological conclusions, and only Einstein[11], a twenty-five year old scientific “lad” could proclaim that time is really shortened (and even it is able to be distorted as he heralded about ten years later in papers of general relativity and after that).

Has the pressure of the Boeotians’ common sense been weakened now, still a century later than the “lad” Einstein’s age and even two than that of the “outsider” Lobachewski? Quite not! On the contrary: it has been reinforced incredibly. How? And why? This would be to be a subject of another paper, rather in the area of sociology and general philosophy of science and thus, rather aside from the subject of the present paper. In the framework of the later, however, one should pay attention just to other embodiments of the metarelation of idempotency and hierarchy called formal completeness[12], along with that in Hilbert arithmetic and utilized for building an only arithmetical proof of FLT:

As Hilbert arithmetic is complementary to the qubit Hilbert space (in turn isomorphic to the separable complex Hilbert space of quantum mechanics under a few quite elementary and technical conditions), one is to discover the same ontological metarelation of formal completeness (which can be investigated even properly philosophically, e.g., in the framework of “scientific transcendentalism” as in other papers: (e.g., Penchev, 2020) in its fundamental untarity underlying even the constitution as well as the possibility of quantum mechanics to be an objective experimental science, fortunately, also a problem already researched in other papers (e.g., Penchev, 2021). Summarizing their conclusions, unitarity implies the conservation of energy conservation in quantum mechanics (Penchev, 2020) and in turn follows from the more general conservation of quantum information allowing for comparing entities quite different according to their place in the hierarchy of energy or respectively, that of spatial dimensions, e.g., an electron (i.e., a microscopic entity), a human being supplied by an apparatus (i.e., a macroscopic entity), and a star or a nebula (i.e., megascopic entities). What is similar to all of them is quantum information being independent of their energies different from each other practically incommensurably.

However, meaning unitarity or the conservation of quantum information in the context of the present paper and its proper subject, one would be to pay attention to the interpretation of FLT (once it has been realized in terms of Hilbert arithmetic) in concepts, theorems and observations of quantum mechanics. After that, a reflection back from quantum mechanics to FLT again could allow for the substitution of the proof of FLT for the case “$n = 3$”, necessary to be granted for the proof by induction from arithmetic papers (such as Kummer, 1847), by another proof, only within the framework of Hilbert arithmetic, and more precisely said, within that of its dual counterpart of the qubit Hilbert space, e.g., such as the Kochen and Specker (1967) theorem about the absence of hidden variables in quantum mechanics. This is only a horizon of the present, first part of the study, but it will be a proper subject of the investigation in the next part, *Part Two* of the research.

Once a reliable bridge for transferring between Hilbert arithmetic (for the proof of FLT by induction) and its dual counterpart of the qubit Hilbert space (for the proof of FLT in the case of “$n = 3$”) would be established in a reliable enough way intended in *Part Two*, it would pioneer the pathway for the interpretation of other theorems in quantum mechanics, e.g., that of Gleason (1957), by FLT as a next horizon (i.e., a kind of meta-horizon, or a horizon of the horizon) of the present paper and intended to be a proper subject of the investigation in *Part Three*.

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[11] Many papers during many years (e.g., Sallent Del Colombo, 2013; Giné, 2010; Hacyan, 2009; Gingras, 2008; Darrigol, 2004; Boi, 1996; Gray, 1995; Miller, 1992; Goldberg, 1967; 1970) pay attention in detail to the deep shared basis or even synchronicity of Einstein’s special relativity and Poincaré’s ideas and publications.

[12] It can be interpreted e.g., as an “interior operator” in relation to an arbitrary category (Castellini, 2011).

2. Fermat's Last Theorem (FLT) by Both Idempotency and Hierarchy

Even a cursory glance on the huge volume of publications devoted directly on FLT demonstrates that it is an exceptionally popular (even one of the most popular claims of all time, especially after the verified proof of Andrew Wiles and corroborated by an authoritative publication in 1995), but proper mathematical statement without any essential philosophical influence or interpretations: rather a mathematical "curio", which turns out to be inexplicably difficult for proving despite its elementary formulation absolutely understandable for a junior high school student.

On the contrary, the present paper claims its fundamental philosophical sense and meaning, therefore forecasting its huge and revolutionary realization and performance even on the general organization of cognition (for which Foucault (1966) coined the term "episteme") in our epoch. Here, FLT is understood as a statement able to distinguish between "Gödel mathematics" corresponding to Modernity, Cartesian dualism, and mathematics opposed to reality, i.e., mathematical models only eventually applicable to empirical and experimental experience and scientific cognition, on the one hand, and "Hilbert mathematics" meaning a future, neo-Pythagorean episteme[13], after which, mathematics, physics (and all science by it), and philosophy are merged in their foundation (rather than separate foundations) and particularly, heralding a world being properly mathematical in its essence just as Pythagoreanism has always insisted, on the other hand[14].

Speaking not so loosely or more precisely, FLT is a theorem provable in Hilbert mathematics, but unprovable in Gödel mathematics[15] for being a statement, which can be demonstrated to belong to the class of Gödel insoluble ones, each of which is related to itself as false: as if claiming for itself to be false just as the "Liar" (from the namesake ancient aporia) states for herself or himself to lie. This can be visualized directly (as below in the text) by a proof that FLT obeys Yablo paradox, and the latter refers to and only to all Gödel insoluble statements.

Particularly, this means that Wiles's proof inferring FLT as an immediate corollary from the Taniyama-Shimura-Weil conjecture or today's modularity theorem (what is the statement proved by Wiles properly) is not valid in the framework of Gödel mathematics since the cited theorem involves arithmetic (by modular forms) and set theory (by elliptic curves) simultaneously and even equating them in a sense. In other words, this is a direct proof (but without a constructive reference to an explicit step in Wiles's proof), that it is out of the scope of standard mathematics grounded on the three "whales": (Peano) arithmetic, (ZFC)[16] set theory, and (Boolean) propositional logic[17]; for example, by the eventual implicit utilization of so called "inaccessible cardinals", admissibly able to be less than (but not equal to) the least infinite cardinal of any countable sets (as certain authors suggest: e.g., McLarty, 2010; 2020)[18].

However, FLT not only belongs to that class of inherently insoluble propositions in Gödel mathematics, but furthermore it embodies the definition of that class by its formulation and structure rather in a proper philosophical consideration than in a rigorous, formal and mathematical meaning: an observation which will be also demonstrated further in the paper.

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[13] Brouwer's "creative subject" as a proper mathematical method (e.g., Niekus, 1987) is remotely related to the motivation of Hilbert arithmetic in the following sense. The concept of "set", including and especially "infinite set" is necessary to postulate any mathematical entity as existent in advance and therefore, an external viewpoint to it. The method of "creative subject" guarantees that about any stage of mathematical proof rather than about the mathematical entity created by the stage at issue, therefore substituting the latter by the former. Hilbert arithmetic achieves the same objective using the second, dual Peano arithmetic. Kikuchi and Kurahashi (2016) introduce concepts of "illusory" and "insane" models of Peano arithmetics, in terms of which Hilbert arithmetic can be also discussed.

[14] Barrow (2011) argues "that there is no reason for us to expect Gödel incompleteness to handicap the search for a description of the laws of nature, but we do expect it to limit what we can predict about the outcomes of those laws". Paraphrasing or generalizing his idea, one may say that Gödel mathematics would correspond to deterministic (non-deterministic) predictions forecast in its framework. Classical physics (unlike quantum mechanics) predicts just deterministically. Collins (1980) suggests a different viewpoint to the limits of physical knowledge from the viewpoint of "Gödel's proof".

[15] One can admit even a rather philosophical conjecture that the great contemporary mathematical problems such as the CMI "Millennium Prize Problems" (e.g., Carlson *et al.* (Eds.) (2006) to clarify their fundamental meaning for mathematics) are soluble in Hilbert mathematics, but insoluble in Gödel mathematics: as FLT in particular.

[16] The restriction to ZFC is not essential since the enumeration (or recursion) of a countable set is only necessary, and the equivalence in different tuples of axioms of set theory is discussed in detail e.g., in Burger (1971).

[17] Berarducci (1990) considers the consistency of the case if propositional logic is substituted by modal logic among those three whales.

[18] McLarty's approach in many papers (e.g., McLarty, 2013, 1993a; 1993b; 1993c and 1988) can be related to the approach in the present paper. So, sharing the statement that Wiles's proof is out of the scope of the standard mathematics based on arithmetic and set theory is not surprising.

The term "Hilbert arithmetic" connotes implicitly Hilbert's program[19] for mathematics to be justified only on arithmetic (of course, and also on propositional logic), but without set theory supposed to be inferable from arithmetic before the Gödel (1931) incompleteness theorems. After Hilbert arithmetic underlies Hilbert mathematics justified in other papers (Penchev, 2021), therefore neutralizing the aftershocks of those theorems, one can realize Hilbert arithmetics to be relevant to the intention or objective of Hilbert's program, but in a way either indirect or very sophisticated and complicated, or simple, but loose and informal. So, that relation of Hilbert arithmetic to Hilbert's program (in fact meaning Peano arithmetic rather than Hilbert arithmetic) needs an extended comment as follows:

The essence of the Hilbert program relates to the complete representation of any infinite set arithmetically granting both set theory and arithmetic to be first-order logic to propositional logic. The main counterargument consists in the Gödel incompleteness theorems, which are furthermore usually reckoned to be its refutation. One can trace the logical tension of set theory and arithmetic back to the relation of the axiom of induction in arithmetic and the axiom of infinity (or equivalent to it) in set theory since they claim the opposite and thus contradictory statements about the same, namely about *all natural numbers in arithmetic* to the *set of all natural numbers in set theory*. All natural numbers are finite according to the axiom of induction, but simultaneously, the set of all natural numbers is infinite according to the set-theoretical axiom of infinity.

The same observation can also demonstrated by the eventual proof of FLT since it should refer to an infinite set of natural numbers, namely, all natural numbers greater than 2. Indeed, all natural numbers greater than 2 are finite in arithmetic (according to the axiom of induction[20]), but nonetheless their set is necessary to be infinite in set theory in virtue of the axiom of infinity. Thus, one reveals in relation to the eventual solution of FLT the same tension explicated as a dichotomy by the Gödel incompleteness theorems: that general solution has to be either incomplete or contradictory sharing the same problem as the Hilbert program.

One must emphasize expressively that the insolubility at issue is valid if and only if both arithmetic and set theory, or more precisely, the axiom of induction and the axiom of infinity are granted to be valid simultaneously. Thus, any proof of FLT involving simultaneously arithmetic and set theory shares the same trouble necessarily. Wiles's proof is just belonging to that class since it infers FLT as a corollary from the modularity theorem (i.e., by the Tanyama-Shimura-Weil conjecture once it has been already proved as a theorem) even only for the formulation of the latter linking the discrete modular forms (in arithmetic) with the continuous elliptic curves (needing set theory for being continuous). In other words, Wiles's proof is necessary to be out of the scope of Gödel mathematics and this statement will be demonstrated once again in Section IV tracing explicitly that FLT is a Gödel insoluble statement by mediation of Yablo's paradox.

A possible corollary from the observation formulated in the previous paragraph is that it is not valid for arithmetic alone, since it refers to the simultaneous use of arithmetic and set theory (as Wiles's proof does). So, if one manages to restrict the utilized tools only to arithmetic (as it would be plausible to the proof claimed, but not demonstrated by Fermat himself), the above trouble would be workable, however that pathway meets a not less problem, which can be illustrated in detail by Carroll's "What the Tortoise said to Achilles" (Carroll, 1895). Or speaking metaphorically, what the Tortoise said to Achilles is valid to any proof of FLT by only arithmetical tools, namely:

Just as Achilles needs set theory to overtake the Tortoise though as a "Wittgenstein ladder", arithmetic as a first-order logic to propositional logic needs set theory again as a "Wittgenstein ladder" for any implication to be accomplishable

[19] Hilbert's program is closest to a kind of neo-Pythagoreanism (relevant to the position of the present paper). Anyway, the link between it and Pythagoreanism is not discussed at all (e.g., Frank, 2009; Feferman, 2008 and 1988; Rathjen, 2005; Zach, 2004 and 2003; Raatikainen, 2003; Grattan-Guinness, 2000; Judson, 1997; Sieg, 1988; Simpson, 1988). Anyway, Velenan (1997) links FLT directly to Hilbert's program. However, Gödel's ontological proof of God's existence in its variants (Hajek, 2011) can be interpreted as relevant to Pythagoreanism; also other mathematical and philosophical views (e.g., Odifreddi, 2011) can be interpreted as implicitly Pythagorean. Jongeneel and Koppelaar (1999) clarifies the way, in which Godel's incompleteness theorems can seves pro and contra AI. LaForte *et al.* (1998) oppose Penrose (2011, for example), who (according to those authors) "claims to prove that Gödel's theorem implies that human thought cannot be mechanized". Indeed, quantum neo-Pythagoreanism is related to "quantum computer" and its interpretation as AI. Lacki (2000) connects the early axiomatizations of quantum mechanics and the continuation of Hilbert's program. The later innovation of Hilbert's program by epsilon, also consistent with Heyting's approach (e.g., Mints, 1977; Stein, 1980; or Visser, 1982) or cut elimination (Mints, 2008) is even more closely related to the direction of the present paper. The Herbrand complexity of cut-elimination (Moser and Zach, 2006) can be related to the wave function as a transfinite number can be interpreted in Hilbert arithmetic.

[20] Indeed "3" is finite; if any "n>3" is finite, "n+1" is also finite; thus the axiom of induction implies that any "n>3" is finite. (Of course, all natural numbers are finite again by the axiom of induction).

(as Carroll demonstrated properly: Penchev (2021). Particularly, set theory is especially necessary for the proof of FLT since it is to "overtake" all natural numbers just as Achilles needs it to overtake the Tortoise, though after overtaking, set theory is not more necessary and can be absolutely abandoned. Speaking figuratively, any proof of FLT including an eventual purely arithmetical one has to "wade between Scylla and Charybdis": that is, between the Gödel dichotomy of "either incompleteness or contradiction", on the one hand, and "What the Tortoise said to Achilles", on the other hand. The former excludes arithmetic and set theory to be used simultaneously, but the latter requires them to be used just simultaneously (as if exemplifying the confusing Hegelian "dialectic discourse" able to go crazy for any logician).

Fortunately, the mediation of propositional logic in both cases allows for the Hegelian "dialectic contradiction" to be substituted by a safe and only alleged contradiction for both not to be utilized simultaneously, or to be complementary to each other (in the sense of Bohr's "quantum complementarity"). Speaking otherwise, the "Witgentein ladder" of set theory can be substituted by a doubled "Witgentein ladder" (respectively, two "Wittgenstein ladders"), however usable not simultaneously, or in other words, complementarily to each other: in fact borrowed from Hilbert arithmetic anyway and therefore demonstrable as a purely arithmetical proof of FLT: thus accomplishable by Fermat himself (as he claimed).

Speaking rigorously and properly mathematically, one needs the bijection of Hilbert arithmetic onto Peano arithmetic, after which the former can be absolutely removed (just as a "Wittgenstein ladder"). Particularly, this means that the bijection at issue is to be absolutely and thoroughly representable into only arithmetical concepts, preferably and plausibly available in Fermat's age and thus accessible for him to have really accomplished the claimed proof of FLT.

Indeed, there exists an elementary axiomatic exchange in Peano arithmetic if it has been extended in advance to be an additive group of all integers rather than the semigroup of all natural numbers. Then, the exchange consists into the counterintuitive postilation that any negative integer is greater than any positive integer. Furthermore, the number "zero" is to be as if doubled as both negative and positive integer so that the "negative zero" to be greater than any integer, but the "positive zero" to be less than any integer therefore hinting at a cyclic group both cuttable and joinable just at the integer "zero".

One can immediately notice that the same approach allows for the purely arithmetical introduction of the integer "infinity" to be dual to "zero" once it has been doubled as above. Really, the usual group of all integers is cut at "infinity" and thus "infinity" is doubled into both positive and negative copies of "infinity". Thus, if one compares the additive group of integers, defined "normally" with their additive group where negative integers are greater than the positive ones, i.e., "non-normally" all integers are available as the same in both cases, and this fact calls for "zero" and "infinity" to be identified analogically therefore introducing an arithmetical "infinity" as the dual counterpart of "zero" and after which the axiom of infinity (in that set theory being purely arithmetical) and the axiom of induction are to be dual to each other in turn.

The last observation removes the tension of the Gödel dichotomy of arithmetic to set theory, choosing for arithmetic to be *complementary* to set theory and thus substituting their "fatal" contradiction with a "safe" proper complementarity. Once weaponed by that rigorous, but only arithmetical understanding of infinity, the bijection of Hilbert arithmetic into Peano arithmetic can be represented exceptionally simply: namely, as the natural bijection of the absolute value of all integers into all natural numbers. The so-defined bijection is absolutely accessible already to Fermat, or in other words, the "Wittgenstein ladder" of Hilbert arithmetic is not necessary any more and it can be removed absolutely in the ultimate, purely arithmetical proof of FLT, thus thoroughly accomplishable by Fermat.

The bijection at issue (which can be called briefly the "absolute value bijection" or "AVB") is necessary for rigorously inferring of the modified modus tollens, valid to FLT, from the usual modus tollens under the AVB condition and will be demonstrated in the next Section III in detail.

One should notice that just AVB (or respectively, the modified *modus tollens*) is what embodies the relation of the axioms of infinity and induction correspondingly (or respectively, that of arithmetic and set theory being different in Gödel mathematics versus Hilbert mathematics) in the very symmetric structure of the Fermat equation (i.e., that after FLT).

Thus, FLT is a statement both: (1) provable in Hilbert mathematics, but not provable in Gödel mathematics; (2) embodying the relation of Hilbert mathematics to Gödel mathematics implicitly in its structure. Revealing that embodiment (though being removed in the ultimate approach in a “Wittgenstein manner”) is necessary for the purely arithmetical proof of FLT and eventually accessible to Fermat and accomplishable by him.

Both those properties of FLT discover its fundamental ontological meaning and philosophical sense to be a “secret key” or “secret code” allowing for transforming the episteme of Modernity (which is our usual one and thus, very familiar to us) into a new episteme, which can be called provisionally, the “episteme of quantum neo-Pythagoreanism”.

Finally, that construction by doubling Peano arithmetic in two dual ways (from an additive semigroup to an additive group that the one doubling is isomorphic to that of all integers and the other one can be considered as a substructure of Hilbert arithmetic) can be generalized a few times more and more widely and mathematically, so that one can realize the philosophical sense of FLT as a fundamental relation of hierarchy and idempotency in the final analysis. Those stages are as follows:

The semigroup of all natural numbers can be doubled also as a multiplicative group: then, not “zero”, but “unit” (as the relevant neutral of that group) is to be bifurcated into the special element greater than any natural number, on the one hand, and the element smaller than any natural number, on the other hand. Respectively, “infinity” being multiplicatively forked into “infinitesimally small quantities” and “infinitesimally great quantities”[21] (unlike “negative infinity” versus “positive infinity” in the former, additive case), turns out to be analogically identified just as in the case of the dual counterpart of the additive group of all integers. Concisely, that generalization considers the structure of a group whether multiplicative or additive to which all natural numbers are a semigroup correspondingly multiplicative or additive.

Then, one can introduce an arbitrary well-ordered (and thus, enumerable by natural numbers), and infinite semigroup with any associative operation. Obviously, a doubling procedure is homomorphic to both cases above: additive and multiplicative. One may conclude that any well-ordered infinite semigroup admitting a bijection to all natural numbers can be doubled in two dual ways already described. Meaning the axiom of choice, any infinite set (independently of its set-theoretical power) supplied by an associative operation and closed to it can be considered, first, as a relevant semigroup, then it can be doubled isomorphically to the corresponding two dual Peano arithmetics belonging to Hilbert arithmetic.

Finally, one can introduce an abstract relation of idempotency so that the procedure of doubling satisfies its formal definition and another one of hierarchy, relevant to well-ordering, and a meta-relation between the relations of idempotency and hierarchy[22]. Following the same, rather philosophical intention, one can call that meta-relation “completeness” meaning the basis (suggestable to be only ontological now) of the way of Hilbert mathematics underlain by Hilbert arithmetic to be complete (in fact, including the world and thus reality, remaining external as to Gödel mathematics, within its scope, i.e., as internal):

Then, idempotency is to be related (now properly philosophically) to the fundamental Cartesian relation of “mind” and the “world” (or “body”), i.e., postulating for them to be idempotent to each other; and hierarchy to be limited only within each of them (particularly, a violation of hierarchy in they are mixed and which can be proclaimed to be a philosophical or even logical fallacy). If one considers the hierarchy of any of them, it will be incomplete to the whole of both mind and world: just this is the fundamental philosophical sense of the Gödel incompleteness from the viewpoint of Hilbert mathematics.

Meaning that maximally wide conceptual framework, the philosophical sense of FLT is to be revealed just by the “metarelation” of completeness to both relations of idempotency and hierarchy. The proof of FLT needs completeness, thus Hilbert mathematics, independently of the fact that it is formulated in terms of a single Peano arithmetics, and in virtue of this: it is shared by both Gödel mathematics and Hilbert mathematics. In other words, though being shared as to the way to be formulated, it can be proved only in Hilbert mathematics since it is an insoluble Gödel statement the proper framework of Gödel mathematics (which admits many of those unlike Hilbert mathematics).

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[21] Rather in the sense of Robinson (1966): as to Leibniz’s original infinitesimal, Christian (1984) elucidates the link between Gödel’s “incompleteness” and other conceptions, on the one hand, Leibniz’s notion of the infinitesimals, on the other hand. Robinson (1956) discusses the conception of complete theories, which can be related to the later one, that of nonstandard analysis.

[22] Indeed, both relations of idempotency and well-ordering (hier called hierarchy) are absolutely rigorously defined in mathematics. However, the text now means their looser or philosophical sense directed to elucidate that of FLT and the eventual option of purely arithmetical proof in virtue of the mentioned already mapping of Hilbert arithmetic into Peano arithmetic and which will be discussed in detail in Section 6.

Anyway, the same observation hints at a possible, "clever and cunning" approach for FLT to be proved only in the framework of Peano arithmetic once Hilbert arithmetic is mapped onto Peano arithmetic so that the proof of FLT in Hilbert arithmetic to be absolutely unambiguously and thoroughly mapped in Peano arithmetic, and Hilbert arithmetic can be removed as a "Wittgenstein ladder" in the ultimate result.

Interpreting the same idea of a "cunning" approach to the proof as to the fundamental philosophical sense of FLT, it would mean that FLT is valid in a kind of the world supposed to be Pythagorean and fundamentally mathematical, however the proof in that Pythagorean world can be mapped absolutely exhaustively in terms of only Peano arithmetic. Thus, one can interpret the "lost proof of Fermat" as a counterfactual artifact (which is to be lost necessarily in the real Cartesian course of history):

Indeed, any mathematician, who has lived or is living in the real course of history, needs that "Wittgenstein ladder" fundamentally inaccessible to him or her for it belongs to the alternative, counterfactual world. Nonetheless, if one suggests that Fermat himself can be related to the pre-bifurcational time, divided disjunctively all mathematicians from the "Wittgenstein ladder" necessary for the proof of FLT, he could yet accomplish that proof (lost in our branch of history as if "necessarily") naively in virtue of the fact that the historical bifurcation at issue would not happened yet until the time in which Fermat had proved FLT.

From the same historical and philosophical perspective (particularly, inferable in a Hegelian conceptual framework), the "cunning approach" consists in the option for Hilbert arithmetic and mathematics realizable in our Cartesian branch of history only in many years after the establishment of quantum mechanics to be utilized as that necessary "Wittgenstein ladder" for restoring the alleged lost proof of Fermat in an absolutely rigorous way able to be "naive" only in virtue of huge wisdom accumulated over the centuries.

One can distinguish two Peano arithmetics even within the structure of Fermat's equation (i.e., that meant by FLT) so that one of them is occupied by the arithmetical variables (usually notated as x, y, z) and the other one, by the arithmetical exponent (usually notated as n). Then, each of both is supplied separately by its own "sub-ladder", however both supplied by Hilbert arithmetic though in two different ways and shortly notated as the modified *modus tollens* or MMT (as to the former Peano arithmetic of and the also modified Fermat descent or MFD[23] (as to the latter Peano arithmetic of x, y, z), accordingly:

"MMT": $(x^{n+1} \rightarrow x^n) \Leftrightarrow [\neg(x^n = y^n + z^n) \rightarrow \neg(x^{n+1} = y^{n+1} + z^{n+1})$ where "x^{n+1}", "x^n" mean correspondingly the propositions: "there exists x^{n+1}" and "there exists x^n". Once that they are interpreted as propositions is emphasized expressively, they can be substituted by the quantifier of existence usually notated as "∃" where the special interpretation of the quantifier as a proposition is represented by brackets; that is:

$$[\exists(x^{n+1}) \rightarrow \exists(x^n)] \Leftrightarrow [\neg(x^n = y^n + z^n) \rightarrow \neg(x^{n+1} = y^{n+1} + z^{n+1})]$$

"MFD": if "FLT" is true for "$n = 3$" and "MMT" is true, the axiom of induction implies for "FLT" to be true for any "n".

The proper mathematical validity of both "MMT" and "MFD" will be discussed in the next Section III, but now one can demonstrate that MMT corresponds to idempotency, and MFD to hierarchy, however the sequence of their use has been exchangled, namely from "idempotency (hierarchy)"[24] (as to Hilbert arithmetic) to "hierarchy (idempotency)"[25] (as to Peano arithmetic).

The justification of that exchange is due to the axiom of choice, meaning only that the whole qubit Hilbert space is equivalent to a single qubit of it. The correctness of that exchange seems to be doubtful at first glance since it refers to the axiom of choice, which does not belong to the axioms of Peano arithmetic and even cannot be added to it at all since it means infinite sets and thus, implicitly the axiom of infinity, which contradicts the axiom of induction in turn included expressively in the axiomatic tuple of Peano arithmetic.

Anyway, one can justify the use of the axiom of choice by a meta-consideration referring to the way how the concept of "Wittgenstein ladder" is to be defined rigorously as to two first-order logics such as arithmetic and set

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23 The modified *modus tollens* (MMT) and the Modified Fermat Descent (MFD) are described in detail in other papers (e.g., Penchev, 2020).

24 This means that hierarchy is used first within each qubit, and after that each qubit is doubled by its dual counterpart.

25 This means a well-ordered (infinite) series of bits, i.e., the class of all Turing machine tapes.

theory and their relation. Speaking loosely, the axiom of choice, itself, is used as a "Wittgenstein ladder" available only to the "Wittgenstein ladder" of Hilbert arithmetic, but removed simultaneously with removing the latter, and thus getting missing in Peano arithmetic, only necessary for the formulation and eventual purely arithmetic proof of FLT.

However, speaking explicitly and constructively, the mapping of Hilbert arithmetic consisting of two dual Peano arithmetics into a single one is what removes both infinity (i.e., all infinite sets) and the axiom of choice, relevant to it (them) canceling the difference of the two dual Peano arithmetics by considering the class of them[26]. That erased "bit" can be interpreted philosophically in a few different ways however meaning always the bit at issue: either "infinity" or "finiteness" (in contemporary mathematics); accordingly, either "body" or "mind" (in Cartesian dualism); either our actual branch of history, in which Fermat's proof of FLT is lost (rather in quotation marks, speaking more precisely: "lost"), or a putative or restorable "naive" state before the bifurcation with the counterfactual branch at issue, during which Cartesian dualism had not appeared and did not exist yet.

Thus, removing the "Wittgenstein ladder" of Hilbert arithmetic is able to remove the axiom of choice, since it belongs to the same "ladder" therefore restoring the pre-Cartesian "paradise" before the "apple of the sinn" of dualism to have "eaten", or without any metaphor, the distinction of infinity versus finiteness had not been articulated, and arithmetic had not been yet restricted to finiteness.

Furthermore, one can consider how MMT and MFD embody correspondingly idempotency and hierarchy in more detail. All centuries-old troubles about FLT and its proof can be synthesized once the present viewpoint of hierarchy and idempotency has been accepted as follows. FLT needs both idempotency and hierarchy to be proved (figuratively speaking, for Achilles to be able to overtake the Tortoise following Carroll's "What the Tortoise said to Achilles": Penchev, 2021), on the one hand, and nonetheless, the same approach implies for FLT to be a Gödel insoluble statement in the framework of both idempotency and set theory, which are necessary for their application just in that sequence: idempotency (hierarchy), on the other hand. In other words, "both hands", just described, generate a real logical and mathematical paradox, which can be called the "paradox of the proof of FLT". Particularly, Wiles's proof is impossible to avoid the same paradox, as this will be rigorously shown in Section IV.

The idea to prove FLT by induction advocated here can be called the solution of Fermat's "lost proof": in the above terms, it substitutes "idempotency (hierarchy)" by "hierarchy (idempotency)" valid only under the axiom of choice, indeed also unjoinable to Peano arithmetic in the framework of both arithmetic and set theory again in virtue of the Gödel dichotomy. However, the axiom of choice is utilized only as a "Wittgenstein ladder": i.e., vanishing by itself as an effect of its own use. Someone might reject its use at the point as a logical fallacy or involving a contradiction, however this is not a fallacy, in fact, for the following:

Properly, the direct contradiction of both use and non-use of the axiom of choice (meant by the metaphor of "Wittgenstein ladder" and which also can be called an "objective, dialectical contradiction" in a Hegelianian discourse) is substituted by the "alleged contradiction" (as it is defined formally and logically e.g., Penchev, 2020) of those "use and non-use" involved so to be complementary rather than contradictory to each other: in other words, the axiom of choice is used only in our contemporary context of arithmetic and set theory. That context vanishes immediately just by virtue of the use of the axiom of choice, and as a side-effect, the axiom of choice, itself, vanishes, too: the context is replaced by its complementary one, that of Fermat's age when arithmetic could refer simultaneously to infinity and finiteness since their disjunctive articulation had not appeared yet.

In other words, the use and non-use of the axiom of choice, able to blow up any logical proof as any direct contradiction, is changed by their "cunning" and safe utilization as complementary to each other: either use (in our epoch) or not-use (in Fermat's epoch). Indeed, the two epochs are complementary to each other just as any two moments of time are divided by a finite interval: they cannot happen simultaneously. Speaking more precisely, the "lost proof of FLT" is a metaphor or an exemplification of the method of complementarity to avoid the direct contradiction of arithmetic and set theory (both necessary for the proof of FLT) and thus one can reveal a solution of the "paradox of the proof of FLT" (as it is involved and described above).

[26] In other words, what is canceled is a bit of information consisting of two oppositions (often granted to be two bits): the first opposition means the bit before and after choice, and the second one, each of both equally probable alternatives after choice. The same bit and two oppositions can be interpreted as the two missing bits of classical information after the "teleportation law" of quantum information (Penchev, 2021).

Once the approach of "hierarchy (idempotency)" has been justified as above (indeed rather sophistically), it can be explicitly and constructively embedded in the alleged "lost proof" of Fermat, which will be the subject of investigation in the next Section III. Concisely, MMT refers to idempotency, and MFD, to hierarchy:

Indeed, MMT means only two levels, and it can be considered as a specific relation of order deprived of transitivity (unlike well ordering called hier "hierarchy" being used in a wider and philosophical context). Namely interpreted as idempotency, it can be proved as in Section III. However, the property of transitivity distinguishes hierarchy (respectively MFD) from idempotency (respectively MMT) not allowing it to be related simultaneously to the same mathematical entity for it would turn out both transitive and non-transitive.

Nonetheless, MMT can be also considered as invariant to transitivity since it refers to two levels, which can be included equally well both in a relation of idempotency and in a relation of hierarchy. Thus, it can be proved by idempotency, but once proved and being the same, to be utilized in the context of hierarchy without any formal and logical contradiction: it remains the same in two complementary contexts, as that of transitivity (the context of its proof) as that of utilization (the global context of the FLT proof by induction). So, it can be seen as the real, mathematical and constructive tool to be unified idempotency and hierarchy (contradicting each other, e.g., in relation of not-transitivity versus transitivity correspondingly) painlessly and consistently as to the purely arithmetical proof of FLT.

Though this is not necessary for that ostensibly restored "lost proof of Fermat", MMT can be interpreted not worse in terms of set theory, or more especially of that "Wittgenstein ladder" of the axiom of choice and allowing for the transition from the contemporary context of both arithmetic and set theory to that of Fermat and being only arithmetical. Indeed, just the axiom of choice links the idempotency of any choice (meaning explicitly that of an infinite set since the choice of a finite set can be accomplished with it following always relevant constructive criteria) and hierarchy to each other in virtue of its equivalence to the well-ordering "theorem"[27].

The way for idempotency and hierarchy to be identified by the axiom of choice can be visualized by a bit of information being definable as a choice between two equally probable alternatives, i.e., naturally idempotent to each other. Indeed, that choice simultaneously orders well the two alternatives as the chosen alternative and the unselected alternative. The axiom of choice provably equivalent to the well-ordering theorem only postulates the same property also to any infinite series of bits.

Now, one can consider the choice between the two levels involved in MMT as a bit of information, which can be even an infinite set if set theory is admissible (which is valid only in our contemporary context, but not in Fermat's). The discussed observation can be expressed, but rather loosely and figuratively so. MMT allows for the axiom of choice to act hiddenly or implicitly even in Fermat's context though being only arithmetic and thus rejecting its explicit use needing infinity (respectively the axiom of infinity).

Once MMT has completed its "dialectical" task in a consistent way (namely to unify idempotency and hierarchy avoiding their direct contradiction by the alternation of its two complementary utilizations: as a conclusion of a proof and as a premise of another proof), MFD, properly following an idea used by Fermat himself, allows for an elementary ultimate proof of FLT (or involving now "hierarchy" as to terms of the present paper):

The main peculiarity of MFD is implied by the exchange of idempotency (hierarchy) featuring Hilbert arithmetic into the "naive" hierarchy (idempotency) only possible in Fermat's age "not yet known of original sin" of infinity. The eventual purely arithmetical proof of FLT needs a bijection (or homomorphism) of Hilbert arithmetic into Peano arithmetic and its opportunity has been visualized above by the trivial mapping of the absolute values of all integers into all natural numbers. However, it seems not to be directly applicable to the arithmetical proof of FLT due to the lack of a well-known relevant logical rule of conclusion (which would be accessible as to Fermat, particularly).

On the contrary, if one means *modus tollens* already utilized in MMT, and furthermore, the method of infinite descent invented by Fermat himself, another homomorphism of Hilbert arithmetics into Peano arithmetic calls to pay attention and after which the set of all natural numbers is the same of both cases of the two dual counterparts of Peano arithmetics involved in Hilbert arithmetic, but being well-ordered oppositely to each other: the "semigroup of transfinite

[27] The well-ordering theorem is equivalent to the well-classification theorem (thus in turn of the axiom of choice) always allowing for a hierarchy of propositions relatable to any classification (Giv, 2015). That approach is relevant to the approach to be proved FLT only arithmetically due to exchanging "idempotency (hierarchy)" into "hierarchy (idempotency)" as far as idempotency is involved in any proposition and its negation.

numbers”[28] starts from “infinity” (speaking loosely) decreasing by a unit as the corresponding function of successor (according to Peano’s axiom), and the “semigroup of finite numbers” starts from the unit increasing by a unit as usual.

The next step is that observation to be “translated into Fermat’s language”, i.e., “without infinity”: then, what remains is that the former semigroup (i.e., that of transfinite numbers in the understanding of our epoch) is only decreasing unlike the latter, usual one: and this corresponds and fits exactly to the structure of MMT linking decreasing with increasing equivalently to each other. Properly, a chain of successive MMT can be well-ordered only increasing according to all natural numbers and thus, implying the applicability of the axiom of induction to FLT (since it is not directly applicable, but only after that very sophisticated construction described in the present paper). One can immediately notice that it will also comprise the twin Peano arithmetics of decreasing natural numbers (starting as if from infinity according to the viewpoint of nowadays). So, *modus tollens* is the relevant tool to run both Peano arithmetics only in virtue of the standard arithmetical induction only in the one direction, that of successive decreasing:

That consideration demonstrates that MMT and MFD are sufficient for FLT and its eventual purely arithmetical proof to be represented exhaustively by idempotency and hierarchy.

3. FLT Proved by Induction Rigorously

Another paper (Penchev, 2020) published a few years ago discusses this. The present section represents only comments to that paper, the objective of which was, first of all, to visualize tenets accessible in Fermat’s age, though in a way not rigorous enough as far as they are “restored” on the base of knowledge available nowadays and sketched during the previous two sections. The intended notices refer to the following questions:

(1) Is MMT (i.e., the “modified *modus tollens*”) valid only to the proof of FLT due to its specific features (or symmetries relevant to Hilbert arithmetic and mentioned in the discussion) until now?

(2) How should one relate MMT and MFD (the “Modified Fermat Descent”) to each other? Or in other words, do they not contradict each other so their simultaneous use in a single proof is inadmissible without generating any logical fallacy, therefore canceling the proof itself?

In a sense, the answer of the latter questions is the key to answer the former one, and thus they need be discussed first. Speaking figuratively, both MMT and MFD only together might represent that kind of a generalized or as if doubled induction able to prove FLT: it can be visualized as transfinite induction consisting of two parts, however not situated successively to each other and therefore constituting a single well-ordering[29], in which any finite natural number is less than any transfinite “natural” number, but *complementing* each other just as in the rigorous meaning of Bohr’s complementarity or as two dual Hilbert spaces in a way to represent transfinite induction in the naive “Fermat arithmetic” still unaware of the “original sin of infinity” and accordingly able to proof FLT in that “paradise”, from which it was not yet “expelled”.

The same construction can be again (as in the previous sections) visualized by the “Wittgenstein ladder” of the axiom of choice, by means of which an infinite set, relevant by its well-ordering (also guaranteed by the axiom of choice) to transfinite induction, can be divided into two disjunctive infinite subset, however now situated to be complementary to each other similar to the two alternatives of a bit of information; that is: the same two alternatives are successive to each other in the former case, and the equivalence of the former and latter cases is supplied by the axiom of choice and the well-ordering “theorem”. Then, one can speak of the two Peano arithmetics of Hilbert arithmetic, however as if they are a “coherent quantum superposition” and indistinguishable from each other in any way yet.

Nonetheless, the proof of FLT needs both and thus, the availability of both can be provided only if they are divided (and thus distinguished, but as complementary to each other and thus preventing their direct contradiction) into two means furthermore distributed into two discernibly different areas: logic (from where *modus tollens* has come) and arithmetic (from where induction by the axiom of induction has come). Indeed, one can see MMT as a second induction

[28] The question “When is a total ordering of a semigroup a well-ordering?” and its answer (Révész, 1990) is quite relevant to that transfinite semigroup.

[29] FLT cannot be proved directly by transfinite induction since it needs the concept of infinity to be formulated and once infinity has been involved even implicitly, FLT turns out to be an insoluble statement in that new and extended framework, which already has included infinity. Anyway, transfinite induction can be interpreted even within Peano arithmetic (e.g., Sommer, 1995) and the approach in the present can be also interpreted in the same way: that is transfinite induction applied indirectly or implicitly within Peano arithmetic.

developing it anti-isometrically (i.e., in the opposite direction) to the usual induction in the normal, "straight" direction since the former originates from the dual Peano arithmetic just anti-isometric to its usual and "normal" counterpart.

Anyway, MMT brings with itself a modification of the reverse induction "decreasing" from infinity to the finiteness of the usual natural numbers and which consists in its beginning from the "other end", the opposite end, that of the usual natural numbers: indeed, that "other end" being interpreted in terms of reverse induction (inherent for the dual Peano arithmetic) is just the "end of infinity". One can think of the two branches of MMT (namely: "$x^{n+1} \to x^n$" is the one branch, and "$[\neg(x^n = y^n + z^n)] \to [\neg(x^{n+1} = y^{n+1} + z^{n+1})]$" is the other one) as belonging to the two complementary Peano arithmetics and dual to each other, however now embodied in the distinguishable, but able to be consistent, arithmetic, and logic applied to the same arithmetic. One need investigated the very sophisticated way for both not to contradict each other and not to generate a trivial logical fallacy by distinguishing the sense (and then, the meaning) of the implication (though notated as the same "$\to$" in both) in each branch separately:

One can notice that the implication connects propositions in the latter case, which is its standard use in propositional logic, but its use in the former case needs a justification, namely the following convention. An implication of mathematical objects (such as "x^{n+1}, x^n" in the case) is to be interpreted as their existence, or that the proposition "There exists x^{n+1}" implies the proposition "There exists x^n". Obviously once that convention is accepted, the use defined by it is unambiguous.

Furthermore, the implication "$x^{n+1} \to x^n$" is a corollary from; "$n + 1 \to n$", which in turn can be considered as an interpretation in terms of propositional logic of the proper arithmetical axiom about the existence of the "function successor". In other words, the well-ordering of implications due to its transitivity and the well-ordering of natural numbers due to the function successor are isomorphic and namely that justifies the implication $x^{n+1} \to x^n$ once the convention at issue has been granted.

Meaning the mentioned above substitution of the two dual Peano arithmetics of Hilbert arithmetic by the natural pair of one arithmetic and one propositional logic, one may say that the branch of "$x^{n+1} \to x^n$" refers to arithmetic in that pair, and the branch of $[\neg(x^n = y^n + z^n)] \to [\neg(x^{n+1} = y^{n+1} + z^{n+1})]$ refers to logic in the same pair. However, their identification in virtue of MMT connects heterogeneous propositions, which are anyway logically commensurable due to the *identity* of logic used in two ways: once as a propositional logic to arithmetic, and twice as a propositional logic to itself (i.e., as a trivial first order-logic to itself[30]).

That consideration is not yet sufficient to verify MMT, which is properly a specific statement valid only in the framework of FLT because of the symmetry of Fermat's equation. In fact, that symmetry embodies by itself the symmetry of Hilbert arithmetic into the structure of Fermat's equation and then into FLT allowing for an only arithmetical proof in which Hilbert arithmetic cannot be utilized explicitly.

What is verified until now can be notated as: "$(x^{n+1} \to x^n) \Leftrightarrow [\neg[x^n = (y+z)^n] \to [\neg[x^{n+1} = (y+z)^{n+1}]$" therefore needing yet to be proved that the right part of the latter statement and the right part of FFT can be in turn equated: a statement quite nontrivial, being false in general, but eventually valid as to FLT utilizing the symmetry of Fermat's equation.

To be equated both $[x^n = (y+z)^n]$ and $(x^n = y^n + z^n)$, on the one hand, and $\neg[x^{n+1} = (y+z)^{n+1}]$ and $(x^{n+1} = y^{n+1} + z^{n+1})$, on the other hand, one needs (for example) the proof of the mutual "orthogonality" of "y" and "z"; that is: any product containing the multiplication "y, z" is necessarily zero: a statement being obviously false if "y" and "z" are arithmetical variables not satisfying any additional conditions.

However, the mediation of Hilbert arithmetic allows for the formulation of those relevant conditions utilizing once again the axiom of choice as a "Wittgenstein ladder" able to be unavailable in the ultimate result once it has been involved in the preliminary stage. Anyway, it is to be meant and used in a rather different way, namely as a relation within a bit of information between the chosen and unchosen alternatives or as the isomorphic relation of the "coherent" state of both before choice, on the one hand, versus either of them, but after choice, on the other hand. Indeed, this relation can be reckoned as an equivalent of that "orthogonality" necessary for any product containing the multiplication of the arithmetical variables "y" and "z" to be zero.

30 For example, propositional logic can be interpreted as a lattice (properly Boolean one) over models of Peano arithmetic (e.g., Schmerl, 2010; 2004 and 1993).

However, the "orthogonality" seems to be absurd as to any two variables in the usual arithmetic since the operation of multiplication is unambiguously defined for any values of them. Nonetheless, it is natural in Hilbert arithmetic if those variables are granted to be dual to each other, i.e., each of them to belong to the one Peano arithmetic different from its dual counterpart, to which is to belong the other variables. Anyway, even that representation borrowed from Hilbert arithmetic is not yet sufficient to represent the structure of Fermat's equation, meaning that two dual Peano arithmetics are anti-isometric to each other. One need introduce the mapping of Hilbert arithmetic into Peano arithmetic as above, after which the orthogonality at issue can be embodied by distinguishing of the variables of "*y*" and "*z*" so that the one to belong to the one kind of arithmetic, and the other one correspondingly, to the other kind of arithmetic.

One may visualize the structure of Fermat's equation substituting the mapping of Hilbert arithmetic into Peano arithmetic by the mapping of the Decart product of Peano arithmetic ("P_1") with itself ("P_2") into itself ("P"), or notated symbolically as: $P_1 \otimes P_2 \to P$. Then, the arithmetical variables "*x*", "*y*", and "*z*" belong correspondingly to: P_1, P_2, P. That is: $(x \in P_1), (y \in P_2), (z \in P)$ if Fermat's equation is notated as above: $x^n = y^n + z^n$. Thus, the "orthogonality" of "*y*" and "*z*" is embodied by the natural "orthogonality" of the argument "P_2" and the function "P".

Anyway, one can question how Fermat himself in his epoch might mean that representation about arithmetical variables "*y*" and "*z*", seeming to be too sophisticated even in our epoch: for example, by their geometrical interpretation as spatial volumes since x^3, y^3, z^3, which can be naturally thought to be the measures of spatial volumes, and x^n, y^n, z^n as their generalizations from the dimension of three (which is natural for a volume) to any dimension. Then, $x^n = y^n + z^n$ can be interpreted as an equation about those generalized volumes, which can be added to each other like homogenous volumetric capacity (e.g., like the sum of the amount of water in different volumes).

Indeed, though that sum of generalized volumes is innate after that interpretation, the multiplication of dimensions of different volumes (such as "*y*" in "y^n" with "*z*" in "z^n" in the case of Fermat's equation) is meaningless and can be granted to be zero just as this is necessary for the only *arithmetical* proof of FLT. In fact, the intervention of that kind of *geometric* interpretation by volumes into a proof intended or alleged to be only arithmetic seems to be not quite correct (though it just brings the necessary "orthogonality" in the proof).

However, one should mean that Descartes's "analytic geometry" belonging to to Fermat's time exploited the same kind of merging algebra (respectively, arithmetic) with geometry again just in virtue of the missing concept of "infinity" and thus, the missing opposition of "infinity" and "finiteness". Thus, the ideas of Hilbert arithmetic (explicitly possible only after Hilbert space able to unify arithmetic and geometry consistently even after the opposition of "finiteness" and "infinity" has been introduced expressively after set theory) can be anyway available in Descartes and Fermat's age though in a naive or unarticulated way:

Indeed, if the opposition of infinity and finiteness is absent, the axiom of induction (or simply, mathematical induction) can be used limitlessly without any distinction between its finite or its "transfinite" (by the term of our epoch) uses. Particularly, Fermat might use induction "transfinitely" in a naive way, i.e., unaware of its distinction to finite induction (a distinction appeared only in the 20th century).

One can utilize the concept of "epoché", coined by Huserl originally to reality, now to infinity to represent in a rigorous way the naive attitude to infinity in Fermat's age, after which the distinction of finitiness and infinity can be absolutely abandoned without any influence on the consideration or logical fallacy in the proof. Even more, one can state that FLT can be proved rigorously only after that "epoché to infinity" and inherent for Fermat just as "speaking in prose" is inherent for all people without being aware of "speaking in prose". On the contrary, a "poetry of infinity" featured contemporary mathematics makes that proof impossible.

4. A Formal Proof of the Insolubility of *FLT* in Gödel Mathematics (Arithmetic and Set Theory)

The idea for an only arithmetic proof of FLT (corresponding to the metaphor of "restoring Fermat's lost proof") relies crucially on its interpretation as a Gödel insoluble statement; at that, as a provable proposition rather than as a more or less loose conjecture or a figurative expression. This is not only a quite different realization, but furthermore entirely aside (even rather back, oppositely) to the mainstream mathematicians' meta-understanding of proof at all as a technical puzzle (i.e., very, very complicated as to that of FLT after it had not been "ordered" for almost five centuries before Wiles's innovation).

Wiles's proof (1995) shares its direction to use more and more sophisticated, newer and newer, more and more "cunning" tools so that the puzzle cannot but be resolved in the final analysis, what properly Wiles managed to demonstrate expressively therefore confirming once again the mainstream mathematicians' conviction about all mathematical problems to be too clever puzzles not needing any change of the dominating (in the case, rather philosophical) "paradigm" and thus unintentionally following the viewpoint of Khun's classical paper (1962).

One can introduce the metaphor that any "wall" in mathematical cognition cannot but be destroyed without any change of general direction applying more and more powerful means. However, Wiles's means need be so powerful, that some "dissidents" (e.g., McLarty, 2010; 2020) admit that such powerful tools can be utilized only changing the general direction, and Wiles did this though unofficially, without posting this "quiet change of paradigm" therefore going out of Gödel mathematics possible only in the framework of both arithmetic and set theory. The present paper tries only to show (as a little above) that Willes's going out is necessary since FLT is insoluble in Gödel mathematics.

Once this can be proved rigorously (which is forthcoming in fact), it can be even utilized for another, only arithmetical proof of FLT, however involving powerful philosophical (or philosophical and mathematical, and logical) tools rather than exceptionally mathematical ones as the mathematical mainstream does as getting natural. Following the mentioned "wall in front of cognition", that philosophical approach seeks for ways to bypass it rather than to "pierce it head-on". Indeed, human beings usually research how to bypass any obstacle that turns out to be difficult enough rather than to "pierce it head-on" by more and more efforts.

Those philosophical means relevant can be represented as a modified *"epoché" to infinity* (rather than the original phenomenological "epoché" to reality and suggested by Husserl himself). In other words, one should abandon the question if any mathematical entity is finite or infinite following the new *"epoché"*. Furthermore, it seems to be natural as to Fermat's age and its naivety or inexperience to infinity, which appeared as an articulated concept only a few centuries later, after Cantor's set theory. Then, one can involve the term of *Fermat arithmetic* just for notating that arithmetic does not discuss infinity whether naively (as Fermat's age supposedly) or for too much experience (as our epoch is inclined to praise and flatter itself).

That is: "Fermat arithmetic" is defined after abandoning the problem about any arithmetic entities (e.g., natural numbers) to be either finite or infinite, or any logically consistent combination of both. So, Fermat arithmetic is to be related to our age, corresponding to the state of too much experience rather than the inexperience of the 17th century about infinity[31]. Anyway, it suggests ideas and hypotheses of how that kind of Fermat arithmetic can be represented thoroughly in the options available in Fermat's age, but those are rather loose conjectures or figurative ways to be introduced more or less popularly "Fermat arithmetic" otherwise absolutely rigorous.

One can immediately notice that the "epoché to infinity" of Fermat arithmetic is sufficient to avoid and prevent at all the Gödel incompleteness. This can be demonstrated literally following the text of his paper (Gödel, 1931) or even, quite simply as here: by elucidating that "epoché to infinity" is able to "cure the root" of the Gödel incompleteness consisting in the direct contradiction of the axiom of induction (in Peano arithmetic) and the axiom of infinity (in ZFC set theory) as to all natural numbers in relation to the set of them. Indeed, if one abandons the question whether all natural numbers (respectively, their set) are finite or infinite, that contradiction cannot appear and consequently, the Gödel incompleteness, too.

Then and as a corollary, FLT is not more an insoluble statement (since that insolubility was due just to the incompleteness of Gödel mathematics meaning the framework of both arithmetic and set theory), and one can prove it expressively just as Fermat had mentioned. Anyway, that proof in Fermat arithmetic turns out to be much more complicated than that in Hilbert arithmetic needing to "restore the virginity" of Fermat's age: indeed sophisticated, but possible. Anyway, it might be simultaneously quite simple in Fermat's own performance, since many problems, which Fermat arithmetic is to resolve, even cannot be formulated in his epoch.

A necessary condition of all those considerations and the proper subject of the present section is the proof that FLT is really an insoluble statement in the framework of both arithmetic and set theory. Speaking loosely, the proof cannot fundamentally resolve FLT as to the set-theoretical complement of arithmetic being finite after the axiom of induction to the set theory being infinite after the axiom of infinity. On the contrary, if one admits an "epoché" to the existence of that complement just as Fermat arithmetic does, FLT is already soluble since the eventual insolubility of FLT cannot be even formulated if that is the case.

[31] One can refer (of course, as a joke) to Lady Bracknell's words from the first act of Oscar Wilde's "The importance of being Earnest": "I have always been of opinion that a man who desires to get married should know either everything or nothing. Which do you know?" Analogically, whoever wishes to prove FLT should know either everything (as our epoch praises itself) or nothing (as our epoch alleges that of Fermat).

The intended proof[32] contains a few crucial points, which will be considered successively in detail further. Those are: (1) FLT implies an inductive scheme as to the exponent ("*n*") in Fermat's equation; (2) that inductive scheme can be interpreted as an exemplification of Yablo's paradox[33], after which an exponent to which FLT is an insoluble statement exists necessarily though it cannot be demonstrated constructively; (3) an antinomic statement after Yablo's paradox applied to (2) is a Gödel insoluble proposition[34]. A few preliminary comments to those three stages are:

As to (1): The inductive scheme about FLT meant in the previous section, to say, in a "cataphatic" way, is exploited now "apophatically". That is: the inductive scheme valid under the only condition of the axiom of induction, is transferred now in the framework of both arithmetic and set theory and where both axioms, the axiom of induction and the axiom of infinity are valid. Once this has been done, one can immediately notice that it satisfies the formal structure of Yablo's paradox.

As to (2): The essence of Yablo's paradox can be granted to be built as an antinomy without self-referentiality and therefore just relevant to FLT because it does not involve any self-referentiality. On the contrary and again following intentions and ideas about the eventual inductive proof of FLT in the previous section, its formulation for any certain natural exponent postpones the proof for the next natural number. Then, this circumstance is not crucial in Fermat arithmetic, but it is fatal in the framework of both arithmetic and set theory, and Yablo's paradox is the relevant tool to be demonstrated expressively.

As to (3): All the previous considerations in the paper are based on the Gödel incompleteness rather than on Yablo's paradox, so the tenets in terms of the latter are to be translated in the language of the former. Nonetheless, the translation itself relies on the relation of idempotency and hierarchy penetrating the present approach. Indeed, the self-referentiality for the Gödel incompleteness can be immediately interpreted as idempotency for example in terms of the "Liar"[35]: the composed statement that "I lie that I lie" means that I do not lie in fact. In other words, idempotency is embedded in the Boolean negation, and self-referentiality transforms into double Boolean negation, therefore involving idempotency as a necessary and even sufficient condition once idempotency is used twice successively.

As to hierarchy in Yablo's paradox, it seems to be obvious. So, the intended in (3) translation of FLT into the Gödel incompleteness is only one more example of the general and presumably, philosophical relation of idempotency and hierarchy used many times above Here are the three stages in mathematical detail:

(1) The necessary inductive scheme involved is to be restricted to the usual induction to "*n*". That is: if one notates definitively $FLT(n) = \neg(x^n = y^n + z^n)$, FLT states: $\forall n > 2, n, x, y, z \in \boldsymbol{N}\colon FLT(n) \rightarrow FLT(n+1)$, which obeys the axiom of induction if one has been proved FLT(3) in advance. As to the present consideration, meaning only the applicability of the standard inductive scheme under relevant additional conditions, FLT(3) is considered to be granted and even out of the scope of this first part of the paper. Indeed, FLT(3) was claimed to be proved by many mathematicians in a more or less well-founded way. One can admit that at least Kumer's one (meaning FLT(3) as a particular case among all regular numbers, or regular prime numbers, introduced by him) is sufficiently flawless to be transferred without any additional comments or notices.

As for Fermat, his proof of FLT(4) is available and well-known (Fermat, 1670). So, one admits that the circumstance that prevented Fermat himself from publishing an eventual inductive proof of the general case is not its absence as to him, but the missing other proof: that of FLT(3). Of course, this is only a loose conjecture.

Though the first part of this paper brackets FLT(3) to be out of its scope for its availability in advance, the involvation of Hilbert arithmetic suggests implicitly another way for its proof closer to quantum mechanics by the

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[32] The idea of that proof is suggested initially in: Penchev (2020) August 25.

[33] Yablo's paradox is not relevant to Fermat arithmetic in virtue of (e.g.) the following observation (Bernardy, 2009). "Consider an infinite sequence of sentences, where any sentence refers to the truth values of the subsequent sentences: if the corresponding function is continuous, no paradox arises." One may say that Fermat arithmetic refers to the "phenomenon" of the function at issue, therefore intentionally not distinguishing continuity from discreteness as to the function.

[34] In fact, this is often stated, but in a slightly different form: by clarifying that the alleged non-circularity of Yablo's paradox can be easily overcome (e.g., Beall, 2001 or Payne, 2015). The "infinitary version" of Yablo's paradox, though designed to overcome its circular reduction (Bringsjord and Heuveln, 2003; Bueno and Colyvan, 2003), can also be interpreted as an analogical tenet as well as that of Sorensen (2011) about "bottomless determination". Furthermore, Cieœliñski and Urbaniak (2013) suggest a direct link between Gödel's incompleteness and Yablo's paradox by means of "diagonalization". Yablo's paradox is related to ù-inconsistency by Ketland (2005). Schlenker (2007) discusses the elimination of self-reference in the context of Yablo's paradox.

[35] Hardy (1995) or Ming (2013) as well as (Priest, 1997) or Sorensen (1998) discuss the relation of Yablo's paradox to that of "Liar" in detail. Kurahashi (2014) clarifies the link between Yablo's paradox and Rosser's proofs of the incompleteness theorems (also related to the "Liar") by means of the Rosser-type formalizations of Yablo's paradox.

mediation of the qubit Hilbert space being in turn complementary to Hilbert arithmetic. Indeed, all proofs of FLT(3), including the cited one of Kumer, are accomplished thoroughly in Peano arithmetic or even involving ones or other elements needing set theory those remaining more or less heterogenous to involving Hilbert arithmetic however rather "stylistically" though being absolutely consistent to it.

That "aesthetic" motivation actuates the second part of the paper, the proper subject of which is just the proof of FLT(3) by the Kochen and Specker (1967) theorem being fundamental for quantum mechanics, and by it, for Hilbert arithmetic. Its idea is to demonstrate by *reductio ad absurdum* that the admission that FLT(3) is false implies the availability of "hidden variables" in quantum mechanics, but rejected just by the Kochen-Specker theorem. In a little more detail: the admission that FLT(3) is false would imply the absolute separability of two qubits however remaining necessarily contextual to each other just as a direct corollary from the Kochen–Specker theorem.

The next stage (2) needs to describe the inductive scheme of FLT as in (1) only in terms of Yablo's paradox, therefore showing that FLT obeys it.

One can notice immediately that FLT is an example of Yablo's paradox if FLT is granted to be the *infinite set* of propositions:

$$"FLT(n \geq 3)\ is\ false";$$
$$"FLT(n \geq 4)\ is\ false";$$
$$"FLT(n \geq 5)\ is\ false";$$
$$\dots$$
$$"FLT(n \geq k)\ is\ false";$$
$$\dots$$

That the scheme[36] is equivalent to FLT is obvious as well as it satisfies literally all the conditions of Yablo's paradox. Consequently, its conclusion that there exists some "*n*" so that the statement "*FLT(n) is false* is both true and false" is necessarily valid. In other words, FLT is an insoluble statement if one means the *infinite set* of all statements gettin the scheme above as the antinomy needs. One has to emphasize expressively that a necessary condition is the *infinite se*t of all those statements to be involved explicitly.

On the contrary, if one considers FLT as a usual inductive scheme obeying only the axiom of induction, but without constituting the infinite set of all statements, i.e., set theory not to be involved, no contradiction appears. Even more, if one extends the discussion even to Fermat arithmetic rather than only to Peano arithmetic, the contradiction at issue does not appear, too, The introduction of Fermat arithmetic suggests that one refrains from making any statement about the existence of the infinite set of all those propositions necessary for the Yablo's paradox to be applied to FLT. If that "epoché" holds, one has to abstain from Yablo's paradox itself for a necessary condition is not satisfied: if a necessary condition is a subject of "epoché", any conclusion needing it is does not imply (anway it can be true in virtue of different premises).

One can show where exactly the infinite set of statements is necessary for Yablo's paradox: it needs statements about *all propositions* for any "*n*" greater than a certain "*k*". If one does not admit statements about all those propositions otherwise than in virtue of the axiom of induction, Yablo's paradox is prevented since those statements about all propositions of that kind cannot be inferred from the axiom of induction just due to which their set is crucially necessary for Yablo's paradox: namely, to surround the axiom of induction because it cannot be applied. Then the conclusion is that *FLT in the framework of both arithmetic and set theory is an insoluble statement due to Yablo's paradox*, but this is false in the framework of arithmetic alone.

The last observation implies a direct corollary in relation to Wiles's proof: it cannot be accomplished in the framework of Gödel mathematics. So: either it is wrong if it is in the framework of Gödel mathematics, or it is out of it if it is correct. The dispute about whether Wiles's proof is within the scope of the standard mathematics (which can be equated to Gödel mathematics) is well known (e.g., McLarty, 2010; 2020). The proponents and opponents seek constructive tenets based on the demonstration of syllogism belonging to that proof, however being too bulky and sophisticated. However

[36] If Yablo's paradox is reformulated in second-order languages (e.g., Picollo, 2013), that formal scheme can be rewritten to them accordingly if need be.

the applicability of Yablo's paradox to FLT resolves that dispute, avoiding that massive volume of syllogisms, but nonetheless proving that FLT is an insoluble statement in the framework of both arithmetic and set theory in a quite direct and even elementary way (as above).

Indeed, the modularity theorem needs both arithmetic and set theory for its formulation and if FLT (as Wiles did) is proved as a corollary (from it), the proof goes beyond Gödel mathematics necessarily. The next section will demonstrate explicitly that FLT is provable within Hilbert arithmetic (which in turn is complementary to Gödel mathematics), and then, one can admit that Wiles's proof involves elements relative to it, but in an unarticulated way. This is only a conjecture and its alternative, namely that Wiles's proof is absolutely independent (though necessarily beyond Gödel mathematics), is also possible.

The final stage (3) demonstrates the logical equivalence of an insoluble statement according to the Gödel incompleteness theorem ("Satz VI" in his original paper in 1931), on the one hand, and an insoluble statement due to Yablo's paradox, on the other hand; that is: both opposite implications between each other. If one considers the general scheme of the bijection of an idempotency and an hierarchy, an insoluble statement in any of both seems to conserve its property of insolubility after the bijection.

Nonetheless, its enumeration (i.e., the relevant bijection to the set of all natural numbers, on the one hand, and the inductive construction of that bijection, on the other hand) turns out to be ambiguous just due to the fundamental incompleteness of arithmetic to set theory. Speaking loosely, the insoluble statement at issue (and thus, the eventual bijection to an insoluble statement in the counterpart scheme) should possess a transfinite number[37] to the set of all natural numbers, but a finite number as to the inductive scheme of the bijection. In other words, its insolubility can be interpreted as follows: it is a soluble statement in the inductive scheme of the bijection, but an insoluble statement in the bijection of the two sets just due to the incompleteness, i.e., due to those elements of the infinite set of all natural numbers which are not natural numbers and which constitute an infinite subset of the set of all natural numbers and which can be called "transfinite numbers" or even: "transfinite natural numbers"[38].

Consequently, what is to be properly proved in the stage (3) is that there exists a bijection (between the Gödel and Yablo schemes) which conserves that ambiguity. In other words, all inductive finite natural numbers according to any of them are also all inductive natural numbers according to the other scheme, and respectively, the set of all transfinite numbers is the same in both cases. This seems to be already obvious because both schemes share the same arithmetic and the same set theory, which implies all natural numbers and the set of all natural numbers to remain the same in both schemes. That proof is sufficient, but it can be supported by a more intuitive (being constructive) visualization of what "happens" to a Gödel insoluble statement after the bijection into Yablo's scheme (i.e., demonstrating that it turns out to be a Yablo insoluble statement) as well as vice versa:

According to the proof above, it conserves its set-theoretical transfinite number as well as its inductive finite number. The Gödel insoluble statement states that a statement possessing a certain Gödel number is false and the insoluble statement at issue is just that statement which possesses that unique Gödel number. Furthermore, the unique Gödel number is assigned by an inductive scheme, which guarantees it to be finite. Nonetheless, that number belongs to the subset of all transfinite numbers which are fundamentally irrepresentable arithmetically and this is what generates the incompleteness of arithmetic to set theory[39].

Thus, one can determine a pair of a finite natural number (e.g., notable as "*n*") and still one, but set-theoretical transfinite number, which in turn can be represented as still one finite natural number (e.g. notable as "*k*") by virtue of the axiom of choice and its equivalence to the well-ordering "theorem". Those "*n*" and "*k*" can be identically conserved after passing into Yablo's scheme as those necessary for constituting an insoluble statement already in its framework.

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[37] Any transfinite number possesses a certain "wave function" as its dual counterpart in the qubit Hilbert space, and thus, a certain relevant probabilistic distribution, the characteristic function of which is the wave function at issue. Thus, that transfinite number (respectively, probability distribution) can be attached to a "universal Chaitin machine" as its halting probability just at the natural number (Calude, 2002: "computably enumerable real" number), to which the transfinite number in the present context corresponds unambiguously. Furthermore, one can construct a model of Peano arithmetic directly on probability distributions (e.g., as in Murakami and Tsuboi, 2003).

[38] The transfinite area (as a result of the Gödel incompleteness) can be investigated even as a special mathematical area with its ontology as by Woodin (2011).

[39] Hartmanis (1982) explains the connection between Gödel numberings and countable sets by "creative sets or the complete recursively enumerable sets"; Visser (1984) discusses "recursively enumerable theories" in an analogical context.

One can notice that "n" and "k" both being natural numbers are to be distinguished by the property "randomness" valid for "k" but not not as to "n" since the latter is determined according to the rigorous rules to generate any Gödel number. On the contrary, "k" is assigned only by the axiom of choice valid to the infinite sets and fundamentally excluding (in general) any permanent rule able to generate a constant (similar to that to generate "n"). Those observations are usually meant by the utilization of functors in a proposition such as: "$\forall n, \exists k$: any arithmetical proposition about both n, k"; and this is the way for both constant Gödel number "n" and random natural number "k" to be transferred from the Gödel scheme into Yablo's one[40], conserving the insolubility of the proposition at issue.

5. The Provability of *FLT* in Hilbert Arithmetic

Until now, Hilbert arithmetic was discussed in relation to FLT as a "Wittgenstein ladder" utilized only as a mean to hint at, or suggest how FLT might be eventually proved in Peano arithmetic or in Fermat arithmetic introduced only by the epoché to infinity in the framework of Peano arithmetic, but on the background of set theory (and Cantor's concept of infinity as actual infinity).

Anyway, one can explicitly express the eventual only arithmetical proof of FLT in Hilbert arithmetic being considered as a generalization of Peano arithmetic. Preliminarily, one is to notice that Hilbert arithmetic, unlike Peano arithmetic, is not either incomplete or inconsistent even with set theory; on the contrary it is a consistent and complete foundation of Hilbert mathematics and relevant to its ontological completeness and consistency (e.g., Penchev, 2021[41]). Thus, FLT is not an insoluble statement in Hilbert arithmetics even only because Hilbert mathematics cannot contain any insoluble statement.

One can trace in detail how Hilbert mathematics manages to release from the Gödel dichotomy of the relation of Peano arithmetic to set theory: "either incompleteness or inconsistency". It postulates a dual, anti-isometric, but isomorphic (respectively, homomorphic) counterpart of Peano arithmetic[42], by which all transfinite natural numbers are enumerated in virtue of the axiom of choice in a "reverse direction" starting "from infinity" as the first element of a well-ordering and "moved back" by a function successor modified to be anti-isometric, such as "$n - 1$" (unlike "$n + 1$" in the "straight" direction utilized by Peano arithmetic).

Obviously, the idea of Hilbert arithmetic is borrowed from the qubit Hilbert space (respectively, the separable complex Hilbert space of quantum mechanics), first of all, to adopt and adapt its completeness provable within itself as the theorems of the absence of hidden variables: already for the objective of the consistent completeness of mathematics. Then, Hilbert arithmetic can be obtained from the qubit Hilbert space by the class of equivalence of each qubit, by which the n^{th} qubit is transformed into the usual n^{th} unit of Peano arithmetic, so that its dual space can be immediately represented by the second dual Peano arithmetic involved by Hilbert arithmetic[43], also as a generalization to the standard, thus single Peano arithmetic.

Then, one can visualize the solubility of any statement in the Gödel scheme (and thus, in the Yablo one, too) assigning both parameters, the certain "n" and the random "k", involved at the end of the previous section to specify any insoluble statement whether in the one or in the other scheme as two usual (i.e., finite rather than transfinite) natural numbers, but belonging to each of both dual Peano arithmetics in Hilbert arithmetic, consequently substituting the fundamental randomness of the one of them by their mutual duality forbidding for them to be considered simultaneously (again an approach borrowed from quantum mechanics). Then and as a corollary, any insoluble statement in the one

[40] Those "n" and "k" can be exemplified in Yablo's scheme applied to FLT as follows. One can assume that there exists an "n" such that "FLT ($m > n$)" is not false. Consequently, "FLT ($m > n + 1$)" is false, but there exists some "k" ($k > n + 1$) so that "FLT ($m > k$)" is not false. Indeed, "n" is certain being determined by the assumption, but "k" is not certain (i.e., it can be interpreted to be random) for being determined only by the necessity of its existence. Goldstein (2006) notices the following. "A syntactically correct number-specification may fail to specify any number due to underspecification. For similar reasons, although each sentence in the Yablo sequence is syntactically perfect, none yields a statement with any truth-value." In other words, "k" is not certain just for the "underspecification".

[41] The approach can be mediately related to "arithmetized completeness theorem" "attributed to Hilbert-Bernays" (Dimitracopoulos, 2001), and furthermore in the general scope of the "arithmetization of logic" (Gautier, 2010). On the contrary, Grzegorczyk (2005) suggests "undecidability without arithmetization" if "the decidability is defined directly as the property of graphical discernibility of formulas" (p. 163).

[42] The dual counterpart of Peano arithmetic can be interpreted as a definable model of Peano arithmetic in a model of Peano arithmetic as this is described by Ikeda and Tsuboi (2007) as well as by Strannegård (1999), also by Murawski (1988) or even in Heyting arithmetic as a "constructive nonstandard model" (Moerdijk and Palmgren, 1997).

[43] The dual Peano arithmetic involved in Hilbert arithmetic can be also interpreted by means of the result provided by Enayat (2008). It can be also situated in the domain of models of Peano's arithmetic (Gaifman, 1976) as an "end-extension".

Peano arithmetic is soluble in its dual counterpart, and anyone is free to use both of them, though alternatively or complementarity, so that any statement is soluble, however just in one of them, but which is absolutely sufficient for Hilbert mathematics not to contain any insoluble statement, including FLT as to the case at issue.

The same universal solubility need be formally represented in terms of Fermat arithmetic featured by its epoché to infinity by involving the following bijection: $[P^{+} \otimes P^{-} \rightarrow P^{0}] \leftrightarrow P$ and where the notations are: "P^{+}" the one dual Peano arithmetic in Hilbert arithmetic and featured by the "positive (n + 1)" function successor; "$\otimes$" Descartes product; "P^{-}" the other dual Peano arithmetic in Hilbert arithmetic and featured by the "negative (n – 1)" function successor; "$\rightarrow P^{0}$" the mapping meant as the "epoché" at issue, that is, into Fermat arithmetic, and "P^{0}" is Fermat arithmetic; and finally, "$\leftrightarrow P$" is the bijection or identity to the usual Peano arithmetic, which is notated by "P".

Then, all Gödel insoluble statements meant by "P^{-}" are substituted by the class of equivalence to "P^{+}" (i.e. any column "n" in the Descartes product is replaced by a single "n", but belonging to "P^{+}"); or speaking loosely and more flowerily being paradoxically, the solubility of a proposition consists in its complete insolubility, i.e., in the class of equivalence of all insoluble prepositions referring to it. For example by the "Liar" antinomy: the class of all "propositions of lies" (such as "I lie"; "I lie that I lie", "I lie that I lie that I lie"; and so on) is postulated to be equivalent to the non-false proposition (such as "I do not lie" or "I say the truth"). That substitution by the relevant class of equivalence is isomorphic to that one generating a unit of Hilbert arithmetic from a certain qubit of the qubit Hilbert space as the class of equivalence of all possible values of it or that of an "empty" qubit.

One is to demonstrate MMT and MFD in Hilbert arithmetic, meaning Fermat arithmetic as for the latter to be used as a "Wittgenstein ladder" to FLT in Peano arithmetic. MFD does not generate any problem for it can be repeated literally in each of separately and thus in the "epoché mapping": $P^{+} \otimes P^{-} \rightarrow P^{0}$.

On the contrary, MMT is available in each of P^{+}, P^{-}, P^{0} only secondarily, i.e., in virtue of the eventual proof of MFD, which divides the variables of x, y, z (if Fermat's equation is notated to be: $x^n = y^n + z^n$) as $y \in P^{+}, x \in P^{-}, z \in P^{0}$ (that convention is more for the sake of clarity because of the idempotency of P^{+} and P^{-}, on the one hand, and the commutativity of "y" and "z", on the other hand). In other words, each variable of x, y, z belongs to just one, but different Peano arithmetic correspondingly, which circumstance allows any arithmetical product containing both "y" and "z" to be equated to zero, and MMT to be inferred immediately from:

$$(x^{n+1} \rightarrow x^{n}) \Leftrightarrow \{\neg[x^{n} = (y+z)^{n}] \rightarrow \neg[x^{n+1} = (y+z)^{n+1}]$$

After that, it appears in Peano arithmetic in virtue of the "epoché bijection" and then, it can be found (as if even initially and inherently also in P^{+}, P^{-}, P^{0} but without utilizing it for the proof of MMT, which allows absolute correctness due to avoiding any "vicious circle"[44]).

The philosophical consideration by the relation of idempotency and hierarchy assists to elucidate how Hilbert arithmetics (embodying that relation by itself: i.e. definitively) allows for FLT to be proved arithmetically (whether with the actual infinity of set theory in Hilbert arithmetic or without it in Fermat arithmetic[45]). Hilbert arithmetic introduces the perfect symmetry of idempotency and hierarchy though traditionally to be illustrated by the idempotency as a metarelation to hierarchy (i.e., not less hierarchy can be considered as a metarelation to idempotency just in virtue of which FLT can be proved even "only" arithmetically). That is what allows for hierarchy and idempotency to be exchanged and it can be illustrated by the equivalent transformation of the model of a quantum computer into a Turing machine accomplishing an infinite calculation by processing an infinite binary tape even by means of a finite binary program[46].

Introducing Hilbert arithmetic allows for the proof of FLT to be made absolutely correctly including for that there exist no insoluble statements in Hilbert mathematics, but first of all, the exchange of idempotency and hierarchy to be justified consistently in the framework of the completeness of Hilbert mathematics, and as to FLT particularly, by the ground of MFD in that exchange. Indeed just this allows for the three variables of Fermat's equation, correspondingly,

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[44] If one passes from Fermat arithmetic to an eventual original proof of Fermat in his age, the avoidance of that vicious circle seems to be much more difficult and that is omitted in the previous publication: (Penchev, 2020).

[45] "Actual infinity" is avoided by Cauchy's "delta-epsilon approach", and one can trace Hilbert's "epsilon" back to it as a relative idea directed to the same objective: i.e., "actual infinity" (Grabiner, 2009). Slater (1994; 1991) discusses epsilon calculus in an analogical context.

[46] The approach can be further continued with Gödel incompleteness (e.g., Shankar, 1994), then with the philosophical problem about the (un)knowability of the world (e.g., Svozil, 2011), on the one hand, or with computer science (e.g., Uspenski, 1994), on the other hand.

to be considered initially as each of them belonging to just one of P^+, P^-, P^0 and the proof without any "vicious circle" by the mediation of Hilbert arithmetic, though it can be removed as a "Wittgenstein ladder" in the final analysis.

6. The Mapping of Hilbert Arithmetic into Peano Arithmetic

The mapping of Hilbert arithmetic into Peano arithmetic defined formally by $[P^+ \otimes P^- \to P^0] \leftrightarrow P$ belongs to a more general class of mappings, which can be called "informational", because of the following observation. It replaces bijectively all the elements of a set into bits of them, i.e., by its informational image. The same kind of formal informational mapings is fundamental for "scientific transcendentalism" to be formulated rigorously (and thus, getting falsifiable) in other papers (e.g., Penchev, 2020).

However, the set, an informational image of which is built, is supplied now by an algebraic structure: that, established by Peano axioms for all natural numbers. Furthermore, it makes sense to be introduced in the context of the eventual, only arithmetical proof of FLT if that kind of mappings is able to distinguish consistently the two algebraic operations defined for all natural numbers, addition and multiplication, in a way not to admit the operation of multiplication as to natural numbers belonging to different arithmetics P^+, P^-, P^0, but simultaneously, the same way is to conserve somehow the operation of addition. In other words, the specific service of that informational mapping as to the case of proving FLT arithmetically consists in the justification of dividing the operation of addition from that of multiplication in the case of Peano arithmetic so that addition is valid including elements belonging to those three different arithmetics, but multiplication is definitively zero in the same case:

For example this is possible, or respectively can be visualized, if the three arithmetics are situated on three orthogonal axes corresponding unambiguously to each of them and furthermore: (1) all operations within each axis are kept to be those in Peano arithmetic; (2) the multiplication of natural numbers belonging to different arithmetics (but only generalizing the case if they belong to a single arithmetic) is defined by the scalar product of the corresponding vectors and thus being definitively zero because of the mutual orthogonality of the axes of the three arithmetics; (3) the addition of natural numbers belonging to different arithmetics (again only generalizing the case if they belong to a single arithmetic is defined by the sum of the modules ("absolute values") of the corresponding vectors. Those conventions are sufficient to guarantee the informational mapping of the three arithmetics into Peano arithmetic to possess the property necessary for proving FLT.

One is to interpret the compose mapping "$[P^+ \otimes P^- \to P^0] \leftrightarrow P$" being in essence and in fact a bijection, but rather extraordinary. Its nonstandard unusualness, speaking loosely or conventionally, might be represented by the rather paradoxical notation of a "2:1 bijection" since "bijection" is a synonym of "1:1 mapping", and then in turn, involving mappings of infinite sets, only to which the immediate logical consistency would be possible. Derivatively, this needs the concept of choice inherent for the axiom of choice in particular.

Indeed, a bit of information as the simplest and elementary choice can quite illustrate that "2:1 bijection", i.e., a 2:1 mapping getting equated to a 1:1 one, which is a usual bijection: any bit before choice means a 1:2 mapping for both alternatives are possible; but the same bit after choice means an unambiguous 1:1 bijection for either single alternative has been chosen. Thus the concept of bit, after equating the states before and after choice, by itself definitely involves the interpretation of "2:1" and "1:1" as the same; and then, information as a quantity whether mathematical or physical, but always measurable by bits, means the same relation of "2:1" and "1:1" just sharing the same quality as its unit of a bit. One can trace back to scientific and even philosophical transcendentalism as well as to the link between them, discussed already in other papers (e.g. Penchev, 2020).

The compose mapping $[P^+ \otimes P^- \to P^0] \leftrightarrow P$ means that fundamental idea linked to Hilbert arithmetic and able to supply mathematics underlain by it (i.e. Hilbert mathematics being grounded on Hilbert arithmetics) with ultimate completeness inferable from itself. The same is necessary for proving FLT arithmetically, and more precisely for proving MMT as to FLT because this means the transposition just of a bit from the meta-position to a position therefore needing idempotency or completeness as a necessary condition for its consistency. However, it has been transferred so that neither idempotency nor completeness (nor Hilbert arithmetic consequently) is necessary anymore and accordingly all can be abandoned as "Wittgenstein's ladder" in favor of Fermat arithmetic being sufficient for proving FLT arithmetically.

Meaning the above notices, one can realize the mapping $[P^+ \otimes P^- \to P^0] \leftrightarrow P$ as an extended tautology, namely; "$\forall A$: $[A \otimes A \to A] \leftrightarrow A$" where "A" can be interpreted as any set as any proposition[47] (conventionally granting for "$\otimes$"

[47] Interpreting loosely, this can be embodied in a quite brief, but less rigorous proof of MMT as to FLT based only on the properties of the relation of identity (but needing why it is valid only to FLT to be additionally justified).

to be whether “conjunction” or “disjunction”); and all P^+, P^-, P^0, P to be defined unambiguously by their unique position (or ordering) in the tautology. (For example, “$\forall A$: $[A \otimes A \rightarrow A] \leftrightarrow A$” can be interpreted particularly to a “proposition of identity”: “$"x = y" \leftrightarrow "x = x"$” after realizing: “$A \otimes A \rightarrow A$” as “$x = y$”).

Then, the former and latter mappings in the framework of the composed mapping at issue need different interpretations: the former is neither a “usual bijection” nor a logical equivalence unlike the latter being both. Therefore, the composed mapping is just what is necessary to embody the idea for “2:1” (the former mapping in the composed one) and “1:1” (the latter mapping) as the same.

Only the rigorous formal proof of MMT needs that rather complicated construction under the convention above. It serves as follows: the three arithmetic variables of Fermat’s equation are distributed unambiguously in the three Peano arithmetics (e.g., as above) so that MMT is provable in that case (and valid only to the variables of Fermat’s equation and possibly to other special cases without any attitude to FLT). The bijection to the usual Peano arithmetic conserves MMT and transforms the three variables of FLT, distributed to belong to the three different arithmetics, to be unified into FLT as usual. Since the other crucial element of the proof, MFD is valid also to a single Peano arithmetic, both MFD and MMT, necessary from the proper arithmetical proof of FLT, are proved to be valid into a single Peano arithmetic, and the proof of FLT to be immediately accomplished.

One can notice that the arithmetical solution by utilizing successively MMT and MFD seems to be too exotic and inexplicable as to why it turns out to resolve the centuries-old puzzle if one seeked for them in Fermat’s age. In fact, the context of their relevance to the problem has become clear nowadays, that is after all troubles about the foundations of mathematics and the completeness of quantum mechanics and within their framework synthesized in “Hilbert arithmetic” now applied to FLT, the combination of MMT and MFD does not seem more to be so extraordinary and ridiculous, even almost “insane”, but quite natural and justified.

Nonetheless, both elements, which, if they had been somehow, but unclearly how, got discovered and caught, would be sufficient for the arithmetic proof of FLT in Fermat’s age. The next section will try to suggest, anyway, an idea how they might be more or less naturally “caught” even in the 17th century by an auxiliary concept of “integer volume” (or “natural volume”) interpreting the arithmetical variable $\forall x, n \in N: f(x,n) = x^n$ as a “*n*-dimensional” volume generalizing variable of the volume of a cube with side length of an integer (i.e., consisting of x^n unit cubes) from “$n = 3$” to the general case of an arbitrary natural number.

One can synthesize concisely the parts of the contemporary mathematical viewpoint, enumerated in different contexts in the paper until now, and which all are necessary to justify the application of both MMT and MFD to the proof of FLT.

First of all, the introduction of Hilbert mathematics based on Hilbert arithmetic is crucial since there exist no insoluble statements in its framework unlike that of Gödel mathematics. In turn, this needs the Gödel incompleteness to be re-released as an independent axiom rather than as a theorem inferable from arithmetic, set theory and propositional logic. However, the negation of the Gödel incompleteness (already) axiom meets common sense’s categorical rejection since it contradicts the fundamental organization of science, cognition and experience in Modernity based on the fundamental opposition of “body” (i.e., the material world) versus “mind” (i.e., the mental world, calling for a kind of quantum neo-Pythagoreanism and a quite different place of mathematics in an absolutely reformed and newly “episteme” coined by Michel Foucault’s conception).

That fundamental change even not only in the paradigm of mathematics, but in philosophy (ontology, epistemology, philosophy of science, etc.) can be compared with the cognitive “earthquake” of Lobachevski’s geometry[48], Einstein’s relativity or quantum mechanics. So, that “instinctive” resistance of the scientific (i.e., not only mathematical) “mainstream” prevents fundamentally any corollaries (though being “gains” for science after the intended philosophical reformation), among which the proof of FLT turns out to be:

Indeed, one can demonstrate that FLT is inherently unprovable in Gödel mathematics after an almost elementary consideration based on Yablo’s paradox and its equivalence to the Gödel incompleteness. In fact, Wiles’s proof of FLT obeys the same insolubility in the framework of standard mathematics, i.e., just Gödel mathematics, in turn obeying the dominating episteme of modernity.

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[48] Boi (1996) discusses a kind of implicit scientific transcendentalism in the history of physics and mathematics and in the context of non-euclidean geometry, on the one hand, and that of philosophical transcendentalism, on the other hand.

So, Wiles's proof can be called a "wise decision in Solomon's manner": it goes beyond the epistemological framework of Modernity because this is unavoidable for anybody wishing to prove FLT, but only as an extremely specialized mathematical result, namely the "modularity theorem" able to link discrete mathematics (i.e., arithmetic) to continuous mathematics (only possible based on set theory) as a really fundamental result but only for mathematics and even only for its most advanced development accessible and understandable by several dozen mathematicians.

So he, whether intentionally or not, wanted to avoid common sense's resistance (for which Gauss utilized the metaphor of the "Boeotians's scream" to explain why he did not publish his results in non-Euclidean geometry[49]) as far as what might get angry the contemporary "Boeotians" is "wrapped in ten veils", the unwrapping of which is accessible only to exceptionally high-qualified professional mathematicians. So, any mathematician can use the modularity theorem, but simultaneously its potential revolutionarity is prevented.

That decision is perhaps wise, but not true. The proof of FLT by itself is far not so sophisticated if one dare go beyond the modern episteme by Hilbert mathematics. Even more, Hilbert arithmetic underlying Hilbert mathematics admits a kind of "projection" into arithmetic as "Fermat arithmetic" definiable by an "epoché to infinity" (analogical to Husserl's epoché to reality, which is borrowed and adapted) and thus pioneering even a bridge to a proof of FLT accessible in Fermat's age (and which can be popularized as restoring the alleged "original proof of Fermat").

So, the conclusion that the proof FLT is rather a philosophical result needing a new worldview than an extremely difficult puzzle only for very, very professional mathematicians. On the contrary, it seems to be almost elementary in Hilbert arithmetic and its real difficulty consists in the courage to be challenged the contemporary "Boeotians", scientific mainstream and even common sense rather than in its technical and conceptual, but only mathematical complexity (now interpretable as a kind of "Solomon's wisdom").

Once FLT has been realized to be an insoluble statement in Gödel mathematics, but soluble in Hilbert mathematics, the structure of Hilbert arithmetic itself, unifying both idempotency and hierarchy in a symmetric way can suggest heuristically the two enough tools for the only arithmetic proof of FLT, namely described already in detail above MMT (for idempotency) and MFD (for hierarchy). While the latter is valid even in a single Peano arithmetic, this is false in general as to MMT needing the mapping "$P^+ \otimes P^- \to P^0$" to be only proved and then "$[P^+ \otimes P^- \to P^0] \leftrightarrow P$" to be utilizable together with MMT, on the one hand, or in the framework of Fermat arithmetic, on the other hand. Hilbert arithmetic is absolutely necessary to deliver MMT in Peano arithmetic as provable, and thus getting removed as a "Wittgenstein ladder": the proved validity of MMT to the case of Fermat's equation turns out to be removed, too and unfortunately: so the only way out, but only at first glance, is it to be postulated as a necessary condition for the proof of FLT in Peano arithmetic alone; or said otherwise, the arithmetical proof of FLT to be reduced to that of MMT.

Fortunately, the next section is able to offer an equivalent of the arithmetical proof of MMT, sufficient for the arithmetical proof of FLT, by excluding the operation of multiplication in Peano arithmetic in a way relevant to FLT and introducing algebraic structure, which is a substructure of Peano arithmetic (but maybe not a true substructure, and thus equivalent to it); furthermore, by the aforementioned generalized "integer volumes": an idea accessible to Fermat or his age.

7. The Proof of FLT in Hilbert arithmetic as a Proof of FLT in Peano Arithmetic

The metaphor of "Wittgenstein ladder" has been utilized as a notation of the method for many parts of the proof of FLT in Hilbert arithmetic to be transferred consistently into Peano arithmetic. However, trouble appears about that method as for MMT since it needs the "Wittgenstein ladder" to be conveyed by itself, therefore generating once again a paradox inherent for self-referentiality, though in a modified, "negative" version. It has to be removed for its transport into Peano arithmetic to be ended, but if it is removed, what is transferred also vanishes since this is itself.

One can reflect on that "paradox of the Wittgenstein ladder transferring itself" as a natural paraphrase of the insolubility and thus fundamental unprovability of FLT in Gödel mathematics as the transport means the relation of Hilbert arithmetic and Peano arithmetic, and the former is an "image" of set theory, by which the Gödel dichotomy about the relation of set theory and arithmetic is restored though expressed a little otherwise.

A way out of the problem, seeming to be a crucial refutation for the proof of FLT in Peano arithmetic once Hilbert arithmetic is utilized as a "Wittgenstein ladder" for it, can be visualized by the positive metaphor (instead the negative

[49] For example in: Rosenfeld (1988) or Schein (1979).

one of the “Wittgenstein ladder”) of the paradoxes about self-referentiality: that of “baron Munchausen who pulls himself out of a mire by his own hair”; for example as follows:

Baron Munchausen could not pull himself, once he had fallen in that situation, because his hand pulling his hair is connected steadily to his body and to his hair in the final analysis in advance. So, if the link between his body and arm was somehow removed, he would manage to pull himself out of the mire. Indeed, one can easily figure an external device similar to an artificial mechanical “arm” controlled by his real hand and able to pull him out of the mire without any contradiction to physical laws since the bodily link has been transformed into an only informational one managing the additional arm without any physical connection with it (more precisely, by a negligible slight connection, but sufficient for the carrier of controlling information).

The metaphor of baron Munchausen can be understood in the case of FLT at issue as removing the interfering mediation of “multiplication”, observing that Fermat’s equation does not contain that operation, but it suggested *implicitly* for the exponential form, since the usual prejudice for an exponential form is it to be inferred as an abbreviated notation of multiplication. However, this is only a representation brought about redundantly and wrongly by the human mind in virtue of the above prejudice; as to Fermat’s equation itself, it is absolutely external and not originating from it:

It can be described exhaustively by two *orthogonal additive* semigroups[50] of natural numbers (i.e., excluding absolutely their multiplicative semigroup complemented by our minds automatically, but groundlessly): namely, the one of all exponents “n”, and the other one, of all values of the variables “x, y, z”. Indeed, one can be convinced that they are orthogonal to each other by means of the following visualization:

“n”

…

n $1^n, 2^n, 3^n, \ldots, x^n, y^n, z^n, \ldots$

…

3 $1^3, 2^3, 3^3, \ldots, x^3, y^3, z^3, \ldots$

2 $1^2, 2^2, 3^2, \ldots, x^2, y^2, z^2, \ldots$

1 $1^1, 2^1, 3^1, \ldots, x^1, y^1, z^1, \ldots$ ....“x, y, z”

One can involve Descartes’s invention (thus accessible to Fermat) of a coordinate system, the ordinate of which depicts the one additive semigroup of natural numbers (that of all exponents in Fermat’s equation), and the abscise of which accordingly, the other semigroup of natural numbers (that of all variables of Fermat’s equation). Both orthogonality of two semigroups and missing multiplication in the representation of Fermat’s equation seem to be verified visually and obviously.

The above visualization as the real algebraic structure underlying that of Peano arithmetic, to which FLT is defined explicitly, is quite accessible to Fermat himself by the auxiliary concept of natural volume (in fact, invented only in this paper, but also possible for Fermat to be able to prove FLT arithmetically, i.e., in the framework so-called Fermat arithmetic as here). Its basis is the observation that Fermat’s descent is easily to be modified to MFD, on the one hand, and MMT in turn can be proved after the idea of “natural volumes” therefore supplying Fermat himself with both tools sufficient for him to prove the theorem.

Indeed, the only obstacle for proving FLT by induction consists in the mixed products of both variables y, z: $y^{\lambda} . z^{v}$; λ, $v = 1, 2, 3, \ldots, n$, and which can be interpreted as the distinction (respectively, subtraction) between the standard distributive law, “$a(b + c) = ab + ac$” and the “distributive law of natural volumes”: “$(b + c)^a = b^a + c^a$”. Indeed, if “natural volume” has been defined as derivative from the usual three-dimensional volume “b^3” after the generalization from “3’ to any natural number, but under the condition for “b” to be also a natural number, the “distributive law of natural volumes” seems to be immediately satisfied by virtue of the concept itself for the mixed products do not make sense because the multiplication of any dimension of the one volume by that of the other volume is obviously meaningless.

In other words, the concept of natural volumes is able to offer a simple and physical interpretation of “Hamilton arithmetic” (a term explained in detail above) originating from everyday human experience, and thus able to serve as a

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50 Kossak *et al.* (1989) discuss a relative approach: the “isomorphism type of the multiplicative semigroup uniquely determines the isomorphism type of the additive semigroup”.

mediation from Hamilton arithmetic, in which MMT is easily provable, to Fermat's age, in which MMT seems to be verifiable as to "natural volumes". Consequently, though only hypothetically, Fermat might be weaponed by all tools enough for proving FLT in Fermat arithmetic.

One can object that, nonetheless, that FLT refers to natural numbers therefore involving multiplication, whether we wish it or not, and thus the above way for excluding multiplication is not correct as to FLT. That objection can be avoided anyway by the following lemma:

***Lemma*:** Any solution of Fermat's equation in Peano arithmetic is a solution in the just defined structure of two additive ("orthogonal") semigroups of natural numbers as above.

Thus, if one proves that there exists no solution of Fermat's equation in that structure of two semigroups (which can be briefly called further "Hamilton arithmetic" for considerations which follow quite soon), this implies that Fermat's equation cannot be resolved in Peano arithmetic under the same conditions. The lemma is already sufficient to be proved FLT in Peano arithmetic alone since:

(1) MMT is valid in Hamilton arithmetic[51] as to FLT in virtue of the validity in it of: $y^n + z^n = (y + z)^n$ as far as one can grant that the condition "$y \,.\, z = 0$" in Peano arithmetic is equivalent to the absence of the multiplication in Hamilton arithmetic;

(2) MFD is valid without any change in Hamilton arithmetic as in Peano arithmetic;

(3) The proof of FLT(3) in Peano arithmetic is a proof of FLT(3) in Hamilton arithmetic in virtue of the lemma;

(4) One can prove FLT in Hamilton arithmetic by induction for $n \geq 3$, and this implies FLT for $n \geq 3$ for Peano arithmetic in virtue of the lemma.

As to the proof of the lemma itself, it seems to be rather trivial, e.g., by *reductio ad absurdum*, i.e., investigating the contradiction of the case of any solution of Fermat's equation in Hamilton arithmetic, which is not a solution in Peano arithmetic.

However, though the lemma is sufficient as to the immediate task to be proved FLT in Peano arithmetic, one can naturally conjecture the much stronger statement that Hamilton arithmetic and Peano arithmetic are equivalent to each other meanwhile making clear the notation of "Hamilton arithmetic" by the allusion to the relation of "Hamilton mechanics" to "Lagrange mechanics" meant under the relation at issue: namely that of Hamilton arithmetic to Peano arithmetic.

The analogy can be traced back to the way in which one constructs Hamilton mechanics by postulating for mechanical quantities of "speed" ("impetus") and "position" to be independent of each other rather than linked as in Lagrange mechanics so that "speed" is the time derivative of "position". Indeed, Peano arithmetic similarly connects the additive semigroup with the multiplicative group as far as the operation of multiplication is derivative from that of addition. On the contrary, one can admit the second (only conditionally namable as "multiplicative") semigroup to be introduced absolutely independently following the pattern of the relation of "speed" ("impetus') and "position" in order to be "divorced" as in Hamilton mechanics,

Further, one can assure immediately that the definition of Hamilton arithmetic above, by two independent semigroups (the one of the exponents "n", and the other, of the variables "x, y, z") satisfies the condition for the two semigroups, to which arithmetic can be reduced, to be divided as getting absolutely independent of each other. However, this would not be surprising because the intention led to Hamilton arithmetic is a model of Hilbert arithmetic to be built in Peano arithmetic (though the conjecture now is that maybe the model at issue has turned out to be identical with Peano arithmetic rather than an true substructure within it). Indeed, Hilbert arithmetic itself is dual to the separable complex Hilbert space of quantum mechanics (by the mediation of the qubit Hilbert space), which in turn can be interpreted to be an embodiment of Hamilton mechanics so that its dual space corresponds to the dual Hamilton variable (e.g., "impetus" to "position", but not less, also vice versa).

That observation about the origin of Hamilton arithmetic, i.e., if one is able to link it to set theory using Hilbert arithmetic, can be continued further, to Peano arithmetic, once the conjecture of their equivalence has been granted in advance (as a little above). One is to mean the duality of "zero and infinity" discussed already in the two first sections, after which the two independent additive semigroups necessary for Hamilton arithmetic to be established are recorded

[51] One can admit the result can be generalized, e.g., to "diagonalizable algebras in Peano arithmetic" (Montagna, 1980).

as the additive group of all integers; that is: the dimensional-like limit of infinity (in the case of Hamilton arithmetic) is substituted by that shared limit of the two, negative and positive, semigroups of all integers[52].

Joining all those transitions to each other successively in virtue of the transitivity of each of them, one can trace back the structure of Peano arithmetic to the additive group of all integers[53], on the one hand, and to Hilbert arithmetic (and thus to set theory), on the other hand, therefore noticing the equivalence of set theory and Peanto arithmetic (oppositely to the Gödel dichotomy about their relation): not surprising again since Hilbert arithmetic has been introduced after the negation of the Gödel incompleteness "theorem" (once it had been interpreted before that to be an axiom distinguishing fundamentally the area of Gödel mathematics from that of Hilbert mathematics).

The eventual equivalence of Hamilton arithmetic with Peano arithmetic (both expressing the same in two alternative ways) in the present context of the inductive provability of FLT can be suggested also by their mutual modelability in the following sense. On the one hand, Hamilton arithmetic allows for a model of Hilbert arithmetic within Peano arithmetic, referring to the inductive proof of FLT. On the other hand and before that, Peano arithmetic itself is a substructure and thus interpretable as a model of Hilbert arithmetic. One can admit that two structures are to be equivalent if each of them is able to contain a model of the other one just as two sets, for each of which one can prove to be a subset of the other one. Anyway, the conjecture of the equivalence of Hamilton and Peano arithmetic (and thus, and Hilbert arithmetic) is not yet proved rigorously, but more probable by that tenet (and other ones above).

The main obstacle in front of the eventual equivalence of Peano arithmetic and Hamilton arithmetic consists in the noncommutativity of the operation of exponentiation unlike that of multiplication, i.e., $a^b \neq b^a$ opposed to $a \, . \, b = b \, . \, a$. Anyway, a future paper can try to avoid it in a relevant "cunning" way by the mediation of Hilbert arithmetic again, based on the analogical noncommutativity of two dual namesake (or "number-sake") qubits and thus, of two dual namesake (or "number-sake") units, each of which belongs to a cerian one of both dual Peano arithmetics belonging to Hilbert arithmetic.

However, that intended proof is not necessary as to FLT since its eventual inductive proof sketched here or the formulation of FLT itself do not need the commutativity in question. One can utilize Husserl's "epoché" (already involved properly mathematically as an "epoché to infinity") again: now as an "epoché to commutativity" (meaning the exponentiation group of Hamilton arithmetic).

In the framework of the same discussion of the hypothesis, one may investigate its ontological corollaries (which can be interpreted as epistemological not worse). So, two options to define the same (i.e. "arithmetic") are granted to be eventually equivalent to each other, namely: (1) by two semigroups with two absolutely independent operations (Hamilton arithmetic); (2) by two semigroups with dependent operations, e.g., such as the usual "addition" and "multiplication" linked by just one distributive law determining which of the two operations is the derivative one (but not vice versa) and this is the case of Peano arithmetic, in which both commutative, both associative laws, but one distributive law can be inferred from the Peano axioms, which postulate only the fundamental operation of an operand, the "function successor". Nonetheless, Peano arithmetic can be determined alternatively (e.g. similar to the way for Boolean algebra) only algebraically, by the two semigroups subordinated to each other by the single distributive law.

Then, the question of what about the distributive law in Hamilton arithmetic is natural after its two semigroups are absolutely independent of each other (and this is a necessary condition for FLT to be proved inductively). One can consider two alternative options:

(1) If it cannot be admitted (e.g., if one insists on it to be that of multiplication to addition as in Peano arithmetic since multiplication is not defined in Hamilton arithmetic), any two arbitrary semigroups might constitute a Hamilton arithmetic relevant to them, if they are unified for any reason. Then, one can transform it into a kind of Peano arithmetic after complementing it by just one distributive law. (Peano arithmetic in turn can be reformed into a Boolean algebra by adding the *second* distributive law dual to the first one).

(2) However, the suggested example (a little above in this section) of Hamilton arithmetic relevant for proving FLT demonstrates just the distributive law for MMT to be proved as valid in Hamilton arithmetic (but being invalid in Peano arithmetic): $(y + z)^n = y^n + z^n$. Thus, the new distributive law is a necessary condition for FLT to be proved, and

52 A much wider mathematica context suggests the automorphism groups of models of Peano arithmetic (e.g., Nurkhaidarov, 2010).

53 One can be curious about the transformation of FLT and its arithmetical proof into the negative and positive semigroups of all integers. At least at first glance though, that interpretation of FLT does not suggest any link to other essential statements different from FLT itself.

thus presumably the conjecture of the equivalence of Peano and Hamilton arithmetic (if or once their operations obey the same algebraic laws) seems to be also conjectable for proving arithmetically (inductively) FLT.

Then, if one has managed to prove FLT in this way, that hypothetical equivalence should be also proved in virtue of being a necessary condition for the proof at issue. In other words, the option (1) above is to be rejected. Meaning that consideration, one is already weaponed enough to trace back and attack the relevant ontological foundations (or respectively, corollaries as being necessary conditions) once FLT is provable inductively, and thus, Hamilton and Peano arithmetic might be equivalent eventually:

8. Interim Conclusion: The Ontological Corollaries from *FLT* by Induction Seen by the Mapping of Provabilities

The interim conclusion intends to connect the present, first part of the paper to the next one. The former means only an arithmetical and inductive proof of FLT under the condition that the proof of FLT for $n = 3$ is granted in advance, and also means Hilbert arithmetic to be a "Wittgenstein ladder" to it. The latter will try to describe an eventual proof of the initial case, i.e., $n = 3$, following the same or analogical ideas, so that they are to be summarized just in order to be applied for that objective in the next part.

The approach to the forthcoming proof for FLT (3) relies on the Kochen-Specker (1967) theorem about the absence of hidden variables including not only the ultimate statement, but also their method for proving it. The link (or hint) to FLT consists in the following observation:

The insolubility of Fermat's equation (including for $n = 3$) can be interpreted as that the solution is at least an irrational number (or eventually more irrational numbers, i.e., more than one). Then, if all solutions (independent of how many) of Fermat's equation already for $n = 3$ are irrational numbers, they can be represented as three qubits x, y, z (corresponding to the three arithmetic variables meant by FLT), which cannot share any *hidden variable* since this would imply a rational solution for it. The Kochen-Specker theorem guarantees as a corollary that those three qubits cannot share any hidden variable, thus stating for the solution (if that exists) to be one or more irrational numbers and consequently, FLT(3) in fact.

The conceptual background for applying the Kochen-Specker theorem relatable to an experimental science such as quantum mechanics and FLT being in the scope of "pure" mathematics and the abstract theory of natural numbers can be justified only in Hilbert mathematics, but seeming to be "ridiculous" in Gödel mathematics granted to be the "mathematics in default" in our epoch. Their link is available in advance in Hilbert arithmetic (in which FLT can be formulated again) being dual to the qubit Hilbert space (to which the Kochen-Specker theorem can refer).

The incommensurability of two (or three qubits) is inferable from the Kochen-Specker theorem, and then, corresponsable to the limit of infinity dividing the two dual anti-isometric copies belonging to Hilbert arithmetic just as the eventual inductive proof of FLT in HIlbert arithmetic (above in the parer) suggests for the variables y, z of Fermat's equation to belong (e.g., "z" to the "straight" Peano arithmetic, and "y" to the reverse one), therefore also divided by the same limit of infinity.

Meaning further the mapping of the proof in Hilbert mathematics in Peano arithmetic by the mediation of arithmetic defined by two absolutely independent operations for the two relevant semigroups (and called "Hamilton arithmetic" in the previous section), the same limit of infinity in Hilbert arithmetic is mapped just onto the independence of the two semigroups of Hamilton arithmetic. However, that mapping does not conserve the idempotency of the two dual Peano arithmetic since only a single distributive law is valid therefore implementing a certain asymmetry between the two semigroups though being independent in Hamilton arithmetics unlike Peano arithmetic involving an additive semigroup and a multiplicative semigroup after the operation of multiplication of the latter is derivative to that ("addition") of the former. That derivativeness is one more corollary from the Peano axioms though they define only the function successor of a single operand.

One can trace back (or further as to the context of the present section) the "spontaneous"[54] violation of the symmetry of idempotency after mapping into the two independent semigroups of Hamilton arithmetic even to the asymmetry of the states before and after choice in the axiom of choice, nonetheless implying the equivalence of the

[54] It can be called "spontaneous" due to the absence of any other reason than the mapping itself. The allusion or connotation to the spontaneous violation of symmetry of the Higgs mechanism is intentional. It hints at the fundamental asymmetry "before and after choich" (or "before and after quantum measurement"), thus both physical and mathematical, as the ultimate reason for mass at rest (and then, the Einstein energetic equivalent of mass) to appear at all. In other words, mass is to be understood as that fundamental quantity corresponding to the only qualitative "time arrow" also meaning the same asymmetry, furthermore by the correlation of time and energy if both are granted to be reversible like all other physical quantities.

axiom of choice and the well-ordering "theorem" though the latter originates only from the state after choice in a sense.

The same observation can be supported by the "conservation of energy conservation" in quantum mechanics (Penchev, 2020) in virtue of unitarity, also in the area of physics (which is correlative or even reducible to that of mathematics after Hilbert mathematics, in turn underlain by Hilbert arithmetic being just relevant to FLT in the present context). On the one hand, the states before and after measurement in quantum mechanics are identified (even as a necessary condition for quantum mechanics to be an objective experimental science). On the other hand, the state after measurement is a true part of the state before measurement (or figuratively: the "half" of it, excluding just the "half" of variables, but continuing to be complete). Both premises imply that any quantum state is infinite (in a mathematical sense relevant to "quantum neo-Pythagoreanism") in order to equate a set with its true subset.

However, the same backdoor of infinity for avoiding the contradiction about any quantum state is to be modified as to Fermat arithmetic by its "epoché to infinity", after which the contradiction itself is not possible since that epoché excludes it to be articulated explicitly. For example and following the same kind of epoché, one can state that it implies a derivative epoché to quantum measurement, respectively to the quantum states before and after measurement. Figuratively, if one involves the concept of the "phenomenon of quantum state" quantum mechanics refers to it explicitly or "experienced", and Fermat arithmetic means an analogical "phenomenon of natural numbers", but implicitly or "inexperienced", i.e., "naively".

However, repeating the mentioned link of the present, first part of the paper to the next, second part by the concept of quantum incommensurability, just it is utilized in the proof of the Kochen-Specker theorem, but now interpreted as the limit of infinity between two dual Peano arithmetics in the framework of Hilbert arithmetic, one is to represent it in two ways in Fermat arithmetic to the relation of Peano arithmetic and Hamilton arithmetic.

On the one hand, this is the equivalence of Hamilton and Peano arithmetic in the framework of Fermat arithmetic and its epoché to infinity, which can be considered furthermore as a trivial (or stronger) form of idempotency (thus restorable from the relation of idempotency, but not trivial, of both dual Peano arithmetics of Hilbert arithmetic). On the other hand, this is the asymmetry of the two semigroups of Hamilton arithmetic due to the availability of one single law of distributivity. So, if the second part (similarly to the first one) tries to translate the tenet, inferred from the separable complex Hilbert space as the Kochen-Specker theorem applied for proving FLT(3), into the language of Fermat arithmetic, both "hands" above might be useful.

One can notice that all versions of proving FLT arithmetically share the same ontology of Hilbert mathematics and the postulate of its completeness, implying as a corollary the inexistence of insoluble statements what FLT turns to be in Gödel mathematics due to Yablo's paradox. In other words, the arithmetical proof of FLT needs the absence of any insoluble statement and only after postulating this FLT is provable, though, only hypothetically, it seems that FLT might be provable also in Gödel mathematics, i.e., being consistent to the existence of insoluble statements, however different from FLT. That observation suggests that FLT is a unique insoluble statement in the following sense: its eventual provability forces the necessary provability of any other insoluble statement. Anyway, one can admit that property is not unique for FLT but also valid for any statement, for which would be provable to be a Gödel insoluble statement.

The formal condition, on which mathematical completeness relies, can be realized ontologically as a fundamental or permanent doubling postulated yet in "scientific transcendentalism" (e.g., Penchev, 2020). Furthemore, one can trace back, starting from the ontological totality postulated by transcendentalism to be the absolute totality, a series îf more and more "special totalities", which can be considered as derivative from each other (or regressively "recurrent" in a sense). So, one can consider an equivalent sequence of mappings (weather defined formally and mathematically or more loosely, in a philosophical manner) so that the next one is summoned to represent the previous two alternatives only into one of them. Both allusions, as to a bit of information or as to Hegel's "synthesis of thesis and antithesis", might be relevant to hint at the recurrent formula of any next mapping to be constructed; or successively in detail:

The ultimate ontological totality of transcendentalism is mapped into the one "half" of it, e.g., into the Cartesian "mind", only within which the episteme of Modernity situates mathematics thoroughly (independently of whether "mind" is granted to be "primary", e.g., as in Platonism, or not), therefore halving mathematics itself into Hilbert mathematics, on the one hand, and Gödel mathematics, on the other hand. If the latter is the case FLT is an insoluble statement. As to the former, the chain of mappings continues bisecting it into Hilbert arithmetic (its foundation) and all the rest of mathematics, in the framework of which FLT seems to be again insoluble.

Already, as to Hilbert arithmetic, FLT therefore getting interpreted as relevant just to the foundations of mathematics is already a soluble statement, for which it is soluble also in Hilbert mathematics as far as it contains Hilbert arithmetic as its ground. Hilbert arithmetic is halved in turn into two Peano arithmetics, dual and anti-isometric to each other, on the one hand, and the qubit Hilbert space (consisting also of two dual counterparts) complementary to the former unity of two dual Peano arithmetics. Now, the two dual Peano arithmetics (rather than the two dual qubit Hilbert spaces) are what is chosen. Then and among those two arithmetics, each of them may be chosen as long as only one of them is chosen.

This is the relevant stage in the series of recurrent mappings, on which the only arithmetical proof of FLT can begin since the concept of arithmetical induction already makes sense and can be involved. As to Hilbert arithmetic properly, that mapping can be depicted formally and mathematically as above as "$[P^+ \otimes\ P^- \to P^0] \leftrightarrow P$" therefore specifying Peano arithmetic. as the next consecutive stage of mappings.

In turn Peano arithmetic is forked again into its standard representation by two semigroups of multiplication and addition so that the former cooperation is derivative and thus dependent on the latter one, on the one hand, and the "Hamilton arithmetic" defined by two absolutely independent semigroups. Finally one can notice the asymmetry of the two semigrup following from the availability of one single distributive law (unlike e.g., Boolean algebra with two distributive laws dual to each other after de Morgan's rule).

One can generalize the observation of that uniform sequence (even recurrent in a sense) so that a certain subseries of it to be relevant to the arithmetic proof of FLT as an ontological basis for it, at that starting from the ultimate totality of transcendentalism in the final analysis, and described above. Simultaneously, this chain can be represented quite formally, by the recurrent formula generating a binary algorithm so that both relevant part and proof of FLT are also reducible to the same formula.

Indeed, one can "stare" at the inductive proof of FLT needing both MMT and MFD from the viewpoint of the above recurrent inference, still more being equally relevant to the ontological basis. It was already considered as an embodiment of the relation of idempotency and hierarchy, also available in both Hilbert arithmetic and the qubit Hilbert space, from which the paper gets a boost in its introduction.

So, one can trace a continuity starting as ontological and philosophical in transcendentalism, and gradually and parallelly transformed mathematically step by step to the endpoint of an only inductive and arithmetic proof of FLT. The next part of the paper will demonstrate how that continuity can be further pursued: first by an eventual proof of FLT(3) inferred from the properties of the qubit Hilbert space (respectively, the traditional complex Hilbert space of quantum mechanics), then, translated into Peano arithmetic by the mediation of Hilbert arithmetic again.

References

Akiyoshi, R. (2009). On a Relationship between Gödel's Second Incompleteness Theorem and Hilbert's Program. *Annals of the Japan Association for Philosophy of Science*, 17(1), 13-19.

Alves, S., Fernández, M., Florido, M. and Mackie, I. (2010). Gödel's System τ Revisited. *Theoretical Computer Science*, 411(11-13), 1484-1500.

Barrio, Å.À. (2010). Theories of Truth without Standard Models and Yablo's Sequences. *Studia Logica*, 96(3), 375-391.

Barrow, J.D. (2011). Gödel and Physics. in: M. Baaz, C.H. Papadimitriou, H. W. Putnam, D.S. Scott. and C.L. Harper (eds.), *Kurt Gödel and the Foundations of Mathematics: Horizons of Truth*, 255-276. Cambridge: University Press.

Bernardi, C. (2009). A Topological Approach to Yablo's Paradox. *Notre Dame Journal of Formal Logic*, 50(3), 331-338.

Beall, J. (2001). Is Yablo's Paradox Non-circular? *Analysis*, 61(3), 176-187.

Berarducci, A. (1990). The Interpretability Logic of Peano Arithmetic. *Association for Symbolic Logic*, 55(3), 1059-1089.

Boi, L. (1996). Les géométries non euclidiennes, le problème philosophique de l'espace et la conception transcendantale; Helmholtz et Kant, les néo-kantiens, Einstein, Poincaré et Mach. *Kant-Studien*, 87(3), 257-289.

Bringsjord, S. and Heuveln, B.V. (2003). The 'Mental Eye' Defence of an Infinitized Version of Yablo's Paradox. *Analysis,* 63(1), 61-70, John Wiley and Sons.

Bueno, O. and Colyvan, M. (2003). Yablo's Paradox and Referring to Infinite Objects. *Australasian Journal of Philosophy,* 81(3), 402-412.

Burger, E. (1971). "Über Rekursionssätze in der Bernays-Gödel-und in der Neumann-Quine-Mengenlehre. *Mathematical Logic Quarterly,* 17(1), 25-33.

Calude, C.S. (2002). "Chaitin Ω numbers, Solovay Machines, and Gödel Incompleteness. *Theoretical Computer Science,* 284(2), 269-277.

Carlson, J., Jaffe, A. and Wiles, A. (eds.) (2006). *The Millennium Prize Problems.* Clay Mathematics Institute; Providence, R.I.: American Mathematical Society, Cambridge, MA.

Carroll, L. (1895). What the Tortoise said to Achilles. Mind, 4(14), 278-280 (reprinted in 1995 in: Mind, 104(416): 691-693).

Castellini, G. (2011). Interior Operators in a Category: Idempotency and Heredity. *Topology and its Applications,* 158(17), 2332-2339.

Christian, C.C. (1984). "Gödel's Contribution to the Justification of Leibniz's Notion of the Infinitesimals. *Historia Mathematica,* 11(2), 215-219.

Cieœliñski, C. and Urbaniak, R. (2013). Gödelizing the Yablo Sequence. *Journal of Philosophical Logic,* 42(5), 679-695.

Collins, II G.W. (1980). Gödel's Proof and the Limits of Physical Knowledge. *Astronomy Quarterly,* 3(10), 89-93.

Darrigol, O. (2004). The Mystery of the Einstein–Poincaré Connection. *Isis,* 95(4), 614-626.

Detlefsen, M. (1986). *Hilbert's Program: An Essay of Mathematical Instrumentalism.* Dordrecht: Springer, Netherlands.

Detlefsen, Ì. (1990). On an aAleged refutation of Hilbert's Program using Gödel's First Incompleteness Theorem *Journal of Philosophical Logic,* 19(4), 343-377.

Dimitracopoulos, C. (2001). On End Extensions of Models of Subsystems of Peano Arithmetic. *Theoretical Computer Science,* 257(1-2), 79-84.

Enayat, A. (2008). A Standard Model of Peano Arithmetic with No Conservative Elementary Extension. *Annals of Pure and Applied Logic,* 156(2-3), 308-318.

Foucault, M. (1966). *Les Mots et les Choses.* Paris: Gallimard.

Franks, C. (2009). *The Autonomy of Mathematical Knowledge: Hilbert's Program Revisited.* University Press, Cambridge.

Feferman, S. (1988). Hilbert's Program Relativized; Proof-theoretical and Foundational Reductions. *Journal of Symbolic Logic,* 53(2), 364-384.

Feferman, S. (2008). Lieber Herr Bernays!, Lieber Herr Gödel! Gödel on Finitism, Constructivity and Hilbert's Program. *Dialectica,* 62(2), 179-203.

Fermat, P. (1670). Diophanti Alexandrini Arithmeticorum libri sex, et De numeris multangulis liber unus. Cum commentariis C.G. Bacheti v.c. and obseruationibus D.P. de Fermat .senatoris Tolosani," in Acessit Doctrinae analyticae inuentum nouum, collectum ex varijs eiusdem D. de Fermat epistolis. Tolosae: Excudebat Bernardus Bosc, è regione Collegii Societatis lesu, M. DC. LXX, 338-339 (source: http://books.google.com books?id=yx9VIgea.CEYC. &.hl=&source=gbs_api1670).

Freyd, P. (1987). Choice and Well-ordering. *Annals of Pure and Applied Logic,* 35(1), 149-166.

Gautier, Y. (2010). *Logique arithmétique: L'arithmétisation de la logique*. Presses de l'Université Laval, Quebec city.

Gaifman, H. (1976). Models and Types of Peano's Arithmetic. *Annals of Mathematical Logic,* 9(3), 223-306.

Gentzen, G. (1935). "Untersuchungen über das logische Schließen. I," *Mathematische Zeitschrift,* 39(2), 176-210.

Gentzen, G. (1935). "Untersuchungen über das logische Schließen. II," *Mathematische Zeitschrift,* 39(3), 405-431.

Giné, J. (2010). Einstein versus Lorentz and Poincaré: Open Questions of Credit. *Physics Essays,* 23(1), 92-96.

Gingras, Y. (2008). The Collective Construction of Scientific Memory: The Einstein-Poincaré Connection and its Discontents, 1905-2005. *History of Science*, 46(1), 75-114.

Giv, H.H. (2015). The Axiom of Choice, Well-Ordering, and Well-Classification. *American Mathematical Monthly*, 122(1), 56-59.

Gleason, A.M. (1957). Measures on the closed subspaces of a Hilbert space. *Indiana University Mathematics Journal*, 6(4), 885-893.

Goldberg, S. (1967). Henri Poincaré and Einstein's Theory of Relativity. *American Journal of Physics*, 35(10), 934-944.

Goldberg, S. (1970). Poincaré's Silence and Einstein's Relativity: The Role of Theory and Experiment in Poincaré's Physics. *The British Journal for the History of Science*, 5(1), 73-84.

Gray J.J. (1995). Poincaré, Einstein, and the Theory of Special Relativity. *The Mathematical Intelligencer*, 17(1), 65-67.

Greenberg, M.J. (1993). *Euclidean and Non-Euclidean Geometries. Development and History* (3rd edition). W. H. Freeman, New York.

Gödel, K. (1931). Über formal unentscheidbare Sätze der Principia mathematica und verwandter. Systeme, I. *Monatshefte der Mathematik und Physik*, 38(1): pp. 173-198.

Goldstein, L. (2006). Fibonacci, Yablo, and the Cassationist Approach to Paradox. *Mind*, 115(460), 867-890.

Grabiner, J.V. (2009). Who Gave You the Epsilon? Cauchy and the Origins of Rigorous Calculus. in: (M. Anderson, V. Katz, R. Wilson) *Who Gave you the Epsilon and Other Tales of Mathematical History*, 5-13. Mathematical Association of America, USA.

Grattan-Guinness, I. (2000). *The Search for Mathematical Roots, 1870-1940: Logics, Set Theories and the Foundations of Mathematics from Cantor Through Russell to Godel*. University Press, Princeton.

Grzegorczyk, A. (2005). "Undecidability without Arithmetization. *Studia Logica*, 79(2), 163-230.

Hacyan, S. (2009). Geometry as an Object of Experience: The Missed Debate Between Poincaré and Einstein. *European Journal of Physics*, 30(2), 337-343.

Hájek, P. (2011). Gödel's Ontological Proof and Its Variant. in: (M. Baaz, C.H. Papadimitriou, H.W. Putnam, D.S. Scott. and C.L. Harper. (Eds.), *Kurt Gödel and the Foundations of Mathematics: Horizons of Truth*, 307-322. University press, Cambridge.

Hardy, J. (1995). Is Yablo's paradox Liar-like?. *Analysis*, 55(3), 197-198, John Wiley and Sons.

Hartmanis, J. (1982). A Note on Natural Complete Sets and Gödel Numberings. *Theoretical Computer Science*, 17(1), 75-89.

Hornby, G.S. (2007). Modularity, Reuse, And Hierarchy: Measuring Complexity by Measuring Structure and Organization. *Complexity*, 13(2), 50-61.

Ikeda, K. and Tsuboi, A. (2007). Nonstandard Models that are Definable in Models of Peano Arithmetic. *Mathematical Logic Quarterly*, 53(1), 27-37.

Jongeneel, C.J.B. and Koppelaar, H. (1999). Gödel Pro and Contra AI: Dismissal of the Case. *Engineering Applications of Artificial Intelligence*, 12(5), 655-659.

Judson, C.W. (1997). Hilbert's Formalism ànd Arithmetization îf Mathematics. *Synthese*, 110(1), 1-14.

Kanckos, A. (2010). Consistency of Heyting Arithmetic in Natural Deduction. *Mathematical Logic Quarterly*, 56(6), 611-624.

Ketland, J. (2005). Yablo's Paradox and ù-Inconsistency. *Synthese*, 45(3), 295-302.

Kikuchi, M. and Kurahashi, T. (2016). Illusory Models of Peano Arithmetic. *Journal of Symbolic Logic*, 81(3), 1163-1175.

Kochen, S. and Specker, E.P. (1967). The Problem of Hidden Variables in Quantum Mechanics. *Journal of Mathematics and Mechanics*, 17(1), 59-87.

Kossak, R., Nadel, M. and Schmerl, J. (1989). A Note on the Multiplicative Semigroup of Models of Peano Arithmetic. *Association for Symbolic Logic*, 54(3), 936-940.

Khun ,T.S. (1962). *The Structure of Scientific Revolutions*. University of Chicago Press, Chicago, IL.

Kummer, E. (1847). "Beweis des Fermat'schen Satzes der Unmöglichkeit von $x^{\lambda} + y^{\lambda} = z^{\lambda}$ für eine unendliche Anzahl Primzahlen λ. in Kummer, Ernst Eduard. Andre Weil (Ed.), *Collected papers, Contributions to Number Theory*. 1, 274-297. Springer, Berlin, Heidelberg, New York.

Kurahashi, T. (2014). "Rosser-Type Undecidable Sentences Based on Yablo's Paradox," *Journal of Philosophical Logic*, 43(5), 999-1017.

Lacki, J. (2000). The Early Axiomatizations of Quantum Mechanics: Jordan, von Neumann and the Continuation of Hilbert's Program. *Archive for History of Exact Sciences*, 54(4), 279-318.

LaForte, G., Hayes, P.J. and Ford, K.M. (1998). Why Gödel's Theorem Cannot Refute Computationalism. *Artificial Intelligence*, 104(1-2), 265-286.

McLarty, C. (1988). Defining Sets as Sets of Points of Spaces. *Journal of Philosophical Logic*, 17(1), 75-90.

McLarty, C. (1993a). Anti-foundation and Self-reference. *Journal of Philosophical Logic*, 22(1), 19-28.

McLarty, C. (1993b). Numbers Can Be Just What They Have To, *Noûs*, 27(4), 487-498.

McLarty, C. (1993c). Category Theory in Real Time. *Philosophia Mathematica*, 2(1), 36-44.

McLarty, C. (2010). What Does It Take To Prove Fermat's Last Theorem? Grothendieck And The Logic of Number Theory. *Bulletin of Symbolic Logic*, 16(3), 359-377.

McLarty, C. (2013). Foundations as Truths Which Organize Mathematics.*The Review of Symbolic Logic*, 6(1), 76-86.

McLarty, C. (2020). The Large Structures of Grothendieck Founded on Finite-order Arithmetic. *The Review of. Symbolic Logic*, 13(2), 296-325.

Ming, H. (2013). Equiparadoxicality of Yablo's Paradox and the Liar. *Journal of Logic, Language and Information*, 22(1), 23-31.

Mints, G.E. (1977). Heyting Predicate Calculus with Epsilon Symbol. *Journal of Mathematical Sciences*, 8(3), 317-323.

Mints, G.E. (2008). Cut Elimination for a Simple Formulation of Epsilon Calculus. *Annals of Pure and Applied Logic*, 152(1-3), 148-160.

Miller, A.I. (1992). Scientific Creativity: A Comparative Study of Henri Poincaré and Albert Einstein. *Creativity Research Journal*, 5(4), 385-414.

Moerdijk, I. and Palmgren, Å. (1997). Minimal Models of Heyting Arithmetic. *Journal of Symbolic Logic*, 62(4), 1448-1460.

Montagna, F. (1980). Interpretations of the First-Order Theory of Diagonalizable Algebras in Peano Arithmetic. *Studia Logica*, 39(4), 347-354.

Moser, G. and Zach, R. (2006). The Epsilon Calculus and Herbrand Complexity. *Studia Logica*, 82(1), 133-155.

Murakami, Ì. and Tsuboi, À. (2003). Expanding the Additive Reduct of a Model of Peano Arithmetic. *Mathematical Logic Quarterly*, 49(4), 363-368.

Murawski, R. (1988). Definable Sets and Expansions of Models of Peano Arithmetic. *Archive for Mathematical Logic*, 27(1), 21-33.

Niekus, J. (1987). The Method of the Creative Subject. *Indagationes Mathematicae (Proceedings)*, 90(4), 431-443.

Nurkhaidarov, E. S. (2010). Interstitial and Pseudo Gaps in Models of Peano Arithmetic. *Mathematical Logic Quarterly*, 56(2), 198-204.

Odifreddi, P. (2011). Gödel's Mathematics of Philosophy. in: M. Baaz, C.H. Papadimitriou, H.W. Putnam, D.S. Scott. and C.L. Harper. (Eds.) *Kurt Gödel and the Foundations of Mathematics: Horizons of Truth*, 299-305, University press, Cambridge.

Payne, J. (2015). The Yablo Paradox: An Essay on Circularity. *History and Philosophy of Logic*, 36(2), 188-190.

Penchev, V. (2021). What the Tortoise Said to Achilles: Lewis Carroll's Paradox in Terms of Hilbert Arithmetic. November 18, *SSRN*, https://papers.ssrn.com/sol3/papers.cfm?abstract_id=3944692 or https://dx.doi.org/10.2139/ssrn.3944692

Penchev, V. (2021). The Symmetries of Quantum and Classical Information. The Resurrected 'Ether' of Quantum Information, June 8, *SSRN*, https://papers.ssrn.com/sol3/papers.cfm?abstract_id=3861105 or https://dx.doi.org/10.2139/ssrn.3861105

Penchev, V. (2021). Hilbert Arithmetic as a Pythagorean Arithmetic: Arithmetic as Transcendental, August 24, *SSRN*, https://papers.ssrn.com/sol3/papers.cfm?abstract_id=3909610 or https://dx.doi.org/10.2139/ssrn.3909610

Penchev, V. (2021). 'Two Bits Less' After Quantum-information Conservation And Their Interpretation As 'Distinguishability/Indistinguishability' And 'Classical/Quantum', July 8, *SSRN*, https://papers.ssrn.com/sol3/papers.cfm?abstract_id=3873123, https://dx.doi.org/10.2139/ssrn.3873123

Penchev, V. (2020). Two Deductions: (1) From the Totality to Quantum Information Conservation; (2) From the Latter to Dark Matter and Dark Energy, October 20, *SSRN*, https://dx.doi.org/10.2139/ssrn.3683658 or https://papers.ssrn.com/sol3/papers.cfm?abstract_id=3683658

Penchev, V. (2020). Quantum-Information Conservation. The Problem About 'Hidden Variables', or the 'Conservation of Energy Conservation' in Quantum Mechanics: A Historical Lesson for Future Discoveries. *SSRN*, October 5, https://papers.ssrn.com/sol3/papers.cfm?abstract_id=3675319 or https://dx.doi.org/10.2139/ssrn.3675319

Penchev, V. (2020). "The Relationship of Arithmetic as Two Twin Peano Arithmetic(s) and Set Theory: A New Glance From the Theory of Information. *SSRN*, August 25, https://dx.doi.org/10.2139/ssrn.3656179 or https://papers.ssrn.com/sol3/papers.cfm?abstract_id=3656179

Penchev, V. (2020). "Fermat's Last Theorem Proved by Induction (and Accompanied by a Philosophical Comment). *SSRN*, August 10https://papers.ssrn.com/sol3/papers.cfm?abstract_id=3648127 or https://dx.doi.org/10.2139/ssrn.3648127

Penchev, V. (2016). A New Reading and Comparative Interpretation of Gödel's Completeness (1930) and Incompleteness (1931) Theorems. *Логико-Философские Штудии*, 13(2), 187-188, https://philpapers.org/rec/PENANR

Penchev, V. (2010). The Insolubility of the First Incompleteness Theorem. Gödel and Hilbert Mathematics. *Philosophical Alternatives*, 19(5), 104-119 (èn Bulgarian: Â. (*Неразрешимост на първата теорема за непълнотата. Гьоделова и Хилбертова математика. Философски алтернативи*). https://philpapers.org/rec/PEN-21

Penrose, R. (2011). Gödel, the Mind, and the Laws of Physics. in: M. Baaz, C.H. Papadimitriou, H. W. Putnam, D.S. Scott. and C.L. Harper, (Eds.), *Kurt Gödel and the Foundations of Mathematics: Horizons of Truth*. 339-360, University Press, Cambridge.

Perloff, M. (1996). *Wittgenstein's ladder: poetic language and the strangeness of the ordinary.* University of Chicago Press, Chicago.

Picollo, L.M. (2013). Yablo's Paradox in Second-Order Languages: Consistency and Unsatisfiability. *Studia Logica*, 101(3), 601-617.

Pozsgay, L.J. (1968). Gödel's Second Theorem for Elementary Arithmetic. *Mathematical Logic Quarterly*, 14(1-5), 67-80.

Priest, G. (1997). Yablo's Paradox. *Analysis*, 57(4), 236-242.

Raatikainen, P. (2003). Hilbert's Program Revisited. *Synthese*, 137(1-2), 157-177.

Rathjen, M. (2005). The Constructive Hilbert Program and the Limits of Martin-Löf Type Theory. *Synthese*, 147(1), 81-120.

Révész, G. (1990). When is a Total Ordering of a Semigroup A Well-ordering?. *Semigroup Forum*, 41(1), 123-126.

Reid, L. (1998). Wittgenstein's Ladder: The Tractatus and Nonsense. *Philosophical Investigations*, 21(2), 97-151.

Robinson, A. (1956). *Complete Theories*. North Holland, Amsterdam.

Robinson, A. (1966). *Non-standard Analysis*. North-Holland, Amsterdam

Rosenfeld, A. (1988). *A History Of Non-euclidean Geometry: Evolution of the Concept of a Geometric Space* (translated by A Shenitzer), New York, etc.: Springer.

Ryan, W. (1978). "Gödel's Second Incompleteness Theorem for General Recursive Arithmetic," *Mathematical Logic Quarterly*, 24(25-30), 457-459.

Sallent Del Colombo, E. (2013). Einstein, Poincaré and Special Relativity. *Lettera Matematica*, 1(1-2), 77-78.

Schein, B.M. (1979). History of Non-euclidean Geometry. The Development of the Concept of Geometric Space. *Historia Mathematica*, 6(4), 460-464.

Schlenker, P. (2007). The Elimination of Self-Reference: Generalized Yablo-Series and the Theory of Truth. *Journal of Philosophical Logic*, 36(3), 251-307.

Schmerl, J.H. (1993). Finite Substructure Lattices of Models of Peano Arithmetic. *Proceedings of the American Mathematical Society*, 117(3), 833-838.

Schmerl, J.H. (2004). Substructure Lattices and Almost Minimal end Extensions of Models of Peano Arithmetic. *Mathematical Logic Quarterly*, 50(6), 533-539.

Schmerl, J.H. (2010). Infinite Substructure Lattices of Models of Peano Arithmetic. *Journal of Symbolic Logic*, 75(4), 1366-1382.

Schwarzer, N. (2020). *The Theory of Everything: Quantum And Relativity Is Everywhere - A Fermat Universe*. Jenny Stanford Publishing, Singapore.

Shankar, N. (1994). *Metamathematics, Machines and Gödel's Proof*. Cambridge: University Press.

Sieg, W. (1988). Hilbert's Program Sixty Years Later. *Journal of Symbolic Logic*, 53(2), 338-348.

Simpson, S.G. (1988). Partial Realizations of Hilbert's Program. *Journal of Symbolic Logic*, 53(2), 349-363.

Slater, B.H. (1991). The Epsilon Calculus and its Applications. *Grazer Philosophische Studien*, 41(1), 175-205.

Slater, B.H. (1994). The Epsilon Calculus' Problematic. *Philosophical Papers*, 23(3), 217-242.

Sommer, R. (1995). Transfinite Induction within Peano Arithmetic. *Annals of Pure and Applied Logic*, 76(3), 231-289.

Sorensen, R. (1998). Yablo's Paradox and Kindred Infinite Liars. *Mind*, 107(425), 137-155.

Sorensen, R. (2011). Bottomless Determination: How Yablo can get Proportionality from Gunk. *Analysis*, 71(4), 627-630, John Wiley and Sons.

Stein, M. (1980). Interpretations of Heyting's Arithmetic: An Analysis by Means of a Language with set Symbols. *Annals of Mathematical Logic*, 19(1-2), 1-31.

Strannegård, C. (1999). Interpretability over Peano Arithmetic. *Association for Symbolic Logic*, 64(4), 1407-1425.

Svozil, K. (2011). Physical Unknowables. in: M. Baaz, C.H. Papadimitriou, H.W. Putnam, D.S. Scott. and C.L. Harper. (Eds.) ,*Kurt Gödel and the Foundations of Mathematics: Horizons of Truth*, 213-252. Cambridge: University Press.

Uspensky, V.A. (1994). Gödel's Incompleteness Theorem. *Theoretical Computer Science*, 130(2), 239-319.

Velenan, D.J. (1997). Fermat's Last Theorem and Hilbert's Program. *The Mathematical Intelligencer*, 19(1), 64-67.

Visser, A. (1982). On the Completeness Principle: A Study of Provability in Heyting's Arithmetic and Extensions. *Annals of Mathematical Logic*, 22(3), 263-295.

Visser, A. (1984). The Provability Logics of Recursively Enumerable Theories Extending Peano Arithmetic at Arbitrary Theories Extending Peano Arithmetic. *Journal of Philosophical Logic*, 13(1), 97-113.

Ware, B. (2015). *Dialectic of the Ladder: Wittgenstein, the "Tractatus" and Modernism*. Bloomsbury, London, New York.

Wiles, A. (1995). Modular Elliptic Curves and Fermat's Last Theorem. *Annals of Mathematics*, 141(3), 443-551.

Wirth C.-P. (2008). Hilbert's Epsilon As An Operator of Indefinite Committed Choice. *Journal of Applied Logic*, 6(3), 287-317.

Woodin H.W. (2011). The Transfinite Universe. in: (M. Baaz, C.H. Papadimitriou, H.W. Putnam, D.S. Scott. and C L. Harper. (Eds.), *Kurt Gödel and the Foundations of Mathematics: Horizons of Truth*, 449-474, University Press, Cambridge.

Yablo, S. (1982). Grounding, Dependence, and Paradox.*Journal of Philosophical Logic*, 11(1), 117-137.

Yablo, S. (1985). Truth and Reflection. *Journal of Philosophical Logic*, 14(2), 297-348.

Yablo, S. (1987). Identity, Essence, and Indiscernibility. *The Journal of Philosophy*, 84(6), 293-314.

Yablo, S. (1992a). Cause and Essence. *Synthese*, 93(3, 403-449.

Yablo, S. (1992b). Mental Causation. *Philosophical Review*, 101(2), 245-280.

Yablo, S. (1993a). Definitions, Consistent and Inconsistent. *Philosophical Studies*, 72(2-3), 147-175.

Yablo, S. (1993b). Is Conceivability a Guide to Possibility? *Philosophy and Phenomenological Research*, 53(1), 1-42.

Yablo, S. (1993c). Paradox Without Self-Reference. *Analysis*, 53(4), 251-252.

Yablo, S. (1997a). Wide Causation. *Noûs*, 31(Supplement s11), 251-281.

Yablo, S. (1997b). Does Ontology Rest on a Mistake?*Aristotelian Society Supplementary Volume*, 72(1), 229-262.

Yablo, S. (2000a). A Reply to New Zeno, 60(266): pp. 148-151.

Yablo, S. (2000b). Abstract Objects: A Case Study. *Noûs*, 36(Supplement s1), 220-240.

Yablo, S. (2002). De Facto Dependence. *The Journal of Philosophy*, 99(3), 130-148.

Yablo, S. (2003). Causal Relevance. *Philosophical Issues*, 13(1), 316-329.

Yablo, S. (2005a). Almog on Descartes's Mind and Body. *Philosophy and Phenomenological Research*, 70(3), 709-716.

Yablo, S. (2005b). Prime Causation. *Philosophy and Phenomenological Research*, 70(2), 459-467.

Yablo, S. (2016). Parts and Differences. *Philosophical Studies*, 173(1), 141-157.

Yagil, G. (1999). Complexity and Hierarchy: A Level Rule. *Complexity*, 4(6), 22-27.

Zach, R. (2003). The Practice of Finitism: Epsilon Calculus and Consistency Proofs in Hilbert's Program. *Synthese*, 137(1-2), 211-259.

Zach, R. (2004). Hilbert's 'Verunglückter Beweis', The First Epsilon Theorem, and Consistency Proofs. *History and Philosophy of Logic*, 25(2), 79-94.

Zheng, X. and Weihrauch K. (2001). The Arithmetical Hierarchy of Real Numbers. *Mathematical Logic Quarterly*, 47(1), 51-65.

Appendix 1

Proof of FLT by Induction in Peano Arithmetic

The appendix contains the concise, only technical exposition of the proof of FLT in Peano arithmetic (without set theory). It consists of the following stages:

Statement 1. Any solution of FLT in Peano arithmetic is a solution of FLT in an auxiliary arithmetic defined as above by:

"n"

…

n $1^n, 2^n, 3^n, \ldots, x^n, y^n, z^n, \ldots$

…

3 $1^3, 2^3, 3^3, \ldots, x^3, y^3, z^3, \ldots$

2 $1^2, 2^2, 3^2, \ldots, x^2, y^2, z^2, \ldots$

1 $1^1, 2^1, 3^1, \ldots, x^1, y^1, z^1, \ldots$ ……………….."x, y, z"

That arithmetic is denoted as "Hamilton arithmetic". It can be interpreted also as an arithmetic of "natural volumes". The multiplication "$a\ .\ b$" (being dependent on "addition") is substituted by "a^b" as the second operation generating a relevant second semigroup (which is the semigroup of multiplication as to Peano arithmetic) therefore implying the corresponding distributive law $(a+b)^c = a^c + b^c$. Furthermore, the second semigroup is absolutely independent of the first one (being the same one of addition in both Peano and Hamilton arithmetics) since "multiplication" is not defined in Hamilton arithmetic at all, but it is necessary for "exponentiation" in Peano arithmetic to be meditatively derivative from "addition"[1].

***A corollary from Statement 1*:** If FLT for any "n" and notatable as FLT(n) does not have any solution in Hamilton arithmetic, the same is valid in Peano arithmetic. Thus, if FLT is proved in Hamilton arithmetic, FLT is proved in Peano arithmetic.

***Statement 2*:** "$[(y^{n+1}+z^{n+1}) \to (y^n+z^n)] \Leftrightarrow [\neg(y^n+z^n) \to \neg(y^{n+1}+z^{n+1})]$ in Hamilton arithmetic. This is true in virtue of *modus tollens*.

***Statement 3* (MMT):** "$(x^{n+1} \to x^n) \Leftrightarrow [\neg(y^n+z^n) \to \neg(y^{n+1}+z^{n+1})]$" in Hamilton arithmetic. This is true as a corollary from *Statement 2* and the distributive law in Hamilton arithmetic, according to which, "$(y + z)^n = y^n + z^n$"

***Statement 4* (MFD):** If FLT(3) and MMT ("Statement 3") are valid, FLT is true in Hamilton arithmetic. This is true since the axiom of induction in Peano arithmetic and referring only to the additive semigroup being the same in both Peano and Hamilton arithmetics allows for it to be transferred in Hamilton arithmetic without any change.

***Statement 5*:** The validity of FLT(3) in Peano arithmetic implies the validity of FLT(3) in Hamilton arithmetic. This is true, because both validities refer to the same subset of the additive semigroup, and therefore being identical in Peano arithmetic and Hamilton arithmetic, namely: $1^3, 2^3, 3^3, \ldots, n^3$. The same subset, to which FLT refers, is the set of all "natural volumes" in a narrow sense (in other words, the interpretation of the at issue subset in Hamilton arithmetic for the case $n = 3$ in a way accessible in Fermat's age), i.e., the geometrical volumes of cubes, the sides of which are all natural numbers.

***Statement 6*:** (proved in advance, e.g., by *Kummer (1847)*). FLT(3) is true (since "3" is a "regular number" in Kummer's meaning).

The sequence of all six statements above implies FLT getting proved only arithmetically and inductively. The option of that kind of proving of FLT is justified as logically and mathematically, as philosophically in detail in the previous corpus of the study. What follows is a rather technical discussion of a few steps necessary for proving correspondingly each of those six statements (and which might seem doubtful and objectionable for somebody):

In *Statement* 1: Any statement or mathematical expression, which does not involve the operation of multiplication explicitly, however remaining the operation of exponentiation absolutely legitimate, is identical in both Peano and Hamilton arithmetics. Thus, Fermat's equation as well as FLT itself are the same in both: this implies that if a triple x, y, z is a solution of Fermat's equation for some exponent n, it is a solution in both arithmetics simultaneously since the equation, for which the triple is a solution, is the same.

In *Statement* 2: This is an identical statement in both Peano and Hamilton arithmetic as two first-order logics to propositional logic. Indeed, *modus tollens* is true in propositional logic, and the mathematical expressions, not referring to multiplication, but only to addition and exponentiation are also identical.

In *Statement* 3: The expressions "x^n" and "x^{n+1}" are to be interpreted as the propositions "There exist 'x^n' in Peano arithmetic or in Hamilton arithmetic" and "There exist 'x^{n+1}' in Peano arithmetic or in Hamilton arithmetic" accordingly. *Statement* 2 and the distributive law in Hamilton arithmetic, "$(y + z)^n = y^n + z^n$", (which, however, is false in Peano arithmetic) implies *Statement 3*. Indeed, that distributive law can be inferred from the same expression in Peano arithmetic under the additional condition valid only in Hamilton arithmetic that any "mixed product" such as "y^λ, $z^{\nu n}$; λ, $\nu = 1, 2, 3, \ldots, n$ does not exist in Hamilton arithmetic and it can be consequently equated to zero, since it does not influence just because it does not exist.

[1] That is: "addition" generates "multiplication", which in turn generates "exponentiation" as to Peano arithmetic. However, "multiplication" is not defined in Hamilton arithmetic. This implies for "addition" and "exponentiation" to be absolutely independent of each other in it.

Appendix 1 (Cont.)

Proof of FLT by Induction in Peano Arithmetic

One can pay attention to the distinction of non-existing from not-defining, if what is not defined can be granted to be uncertain rather than certain, but as zero. What is not defined in mathematics is usually accepted not to exist, but this is to be formulated more precisely: the "mixed product" as above cannot be defined fundamentally in Hamilton arithmetic, since its inexistence is an axiomatic property of Hamilton arithmetic (without which, it is not more "Hamilton arithmetic") rather than not to be defined however being definiable in principle. Consequently, the "mixed product" is equivalent just to zero in Hamilton arithmetic, but not uncertain.

In *Statement* 4: The axiom of induction is a valid statement in Hamilton arithmetic, too, since the axiom of induction involves only "addition" (i.e., does not utilize "multiplication"), and thus it remains the same as a statement which does not include "multiplication". The axiom of induction being a valid statement in Hamilton arithmetic implies *Statement 4* itself.

In *Statement* 5: The validity of FLT(3) in Peano arithmetic implies for it to be true in Hamilton arithmetic again in virtue of the fact that it does not involve "multiplication" (but only "addition" and "exponentiation"). Furthermore, it refers to a single "row" in Hamilton arithmetic, i.e to a certain exponent, namely "3", thus not needing the simultaneous consideration of more than one row (a problem being neglectable in the framework of FLT alone).

In *Statement* 6: At last, many mathematicians claimed that they proved FLT(3). Anyway, Kummer's proof in 1847 is chosen to be cited since it is absolutely rigorous even in contemporary criteria.

Appendix 2

Proof of FLT by Induction in Hilbert Arithmetic

The proof of FLT in Peano arithmetic (as in *Appendix 1*) can be considered as a proof in Hilbert arithmetic after the suggested "epoché to infinity", i.e., in Fermat arithmetic. Indeed, the insolubility of FLT is valid only if Peano arithmetic and set theory are granted together in advance and thus the Cantor "actual infinity" along with all natural numbers of Peano arithmetic, the finiteness of which follows immediately from the axiom of induction (being invalid in set theory as far as it contradicts the axiom of infinity, e.g. in ZFC). Consequently, the insolubility at issue cannot appear if one does not involve "actual infinity" or set theory since this implies the "abstinence from judgment" meant by "epoché to infinity".

However, one can consider the proof of FLT in Hilbert arithmetic in a wider sense, since just Hilbert arithmetic serves a "Wittgenstein ladder" or as a heuristic pathway for the proof of FLT in Peano arithmetic by a consistent method (rather than suddenly and unexpectedly as if mathematicians "trick up their sleeve"). That approach involves two notices as specific for the proof of FLT in Hilbert arithmetic:

1. The one refers to the mapping "$[P^+ \otimes P^- \to P^0] \leftrightarrow P$" as a "nonstandard bijection" described in detail above. It is not necessary to be "retold" only in terms of Peano arithmetic and avoiding any explicit address to Hilbert arithmetic (e.g. for being inexistent in Fermat's age). So, the need to paraphrase it relevantly drops out and one can utilize directly the proper terms of Hilbert arithmetic. Indeed, the pair P^+, P^- can generate immediately a Hamilton arithmetic by themself, being divided from each other by the limit of infinity (rather than by that of zero and described in detail in the first two sections). Hamilton arithmetic in relation to Peano arithmetic can be only postulated e.g. as in *Appendix I*, though Hilbert arithmetic is what might hint at its idea.
2. The proof of MMT can be accomplished without any explicit reference to Hamilton arithmetic restricting itself only to the properties of the nonstandard bijection:

   "$[P^+ \otimes P^- \to P^0] \leftrightarrow P$".

Indeed, one can right equate the mixed product "$y\ .\ z$" to zero if the case is either $y \in P^+; z \in P^-$ or vice versa, but keeping both operations of addition and exponentiation just as this is necessary for MMT to be proved in the discussed case. Properly, FLT is to be proved for that case ($y \in P^+; z \in P^-$) analogically to the initial proof of FLT only to Hamilton arithmetic meaning in fact Peano arithmetic as in Appendix 1. Then and similarly, any solution in P is a solution in the case "$y \in P^+; z \in P^-$" (or vice versa), and *modus tollens* implies FLT in the general case.

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