https://doi.org/10.51483/IJPAMR.2.1.2022.49-64 ISSN: 2789-9160

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Publisher's Home Page: https://www.svedbergopen.com/

Research Paper Open Access

Theory of Pricing in Stochastic Financial Models – Continuous Tim

Hamdin Alafif[1,2*]

[1]Tabuk University, Tabuk 47512, Saudi Arabia. E-mail: hjumaa@ut.edu.sa
[2]Khartoum University, Khartoum, Sudan.

Article Info

Volume 2, Issue 1, April 2022
Received : 21 August 2021
Accepted : 17 February 2022
Published : 05 April 2022
doi: 10.51483/IJPAMR.2.1.2022.49-64

Abstract

In this manuscript we formulate the basic postulate of the Heath-Jarrow-Merton approach and investigate the existence and uniqueness of the Heath-Jarrow-Merton model. We examine the general Heath-Jarrow-Merton set-up and the Gaussian Heath-Jarrow-Merton set-up respectively and also, we present some examples of the Heath-Jarrow-Merton model for the different types of the volatility structure.

Keywords: *Pricing, Heath-Jarrow-Merton approach, Stochastic Financial Models, Volatility*

1. Introduction

1.1. Modeling Principles

In most of the models which have already studied, the interest rate was assumed to be constant. In the real world, it is observed that the loan interest rate depends both on the date t of the loan emission and the date T of the end or 'maturity' of the loan.

Essentially, the issue is to price bond options. We call 'zero-coupon bond' a security paying one dollar at a maturity date T and we note $B(t, T)$ the value of this security at time t. Obviously we have $B(T, T) = 1$ and in a world where the future is certain.

$$B(t, T) = e^{-\int_t^T r(s)ds}$$

For an uncertain future, we must think of the instantaneous rate in terms of a random process: between times t and $t + dt$ it possible to borrow at the rate $r(t)$ (in practice it corresponds to a short rate, for example the overnight rate). To make the modeling rigorous, we shall consider a filtered probability space $\left(\Omega, \mathcal{F}_t, P, (\mathcal{F}_t)_{0\le t\le T}\right)$ and will assume that the filtration $(\Phi_t)_{0\le t\le T}$ is the natural filtration of a standard Brownian motion $(W_t)_{0\le t\le T}$ and $\mathcal{F}_T = \mathcal{F}$.

We introduce a so-called 'riskless' asset, whose price at time t is given by

* *Corresponding author: Hamdin Alafif, Tabuk 47512, Saudi Arabia. E-mail: hjumaa@ut.edu.sa*

$$B_t = e^{\int_0^t r(s)ds}$$

where $(r(t))_{0 \le t \le T}$ is an adapted process satisfying $\int_0^T |r(t)| dt < \infty$, almost surely. It might seem strange that we should call such an asset riskless since its price is random; we will see later why this asset is less 'risky' than the others. The risky asset here are the zero-coupon bonds with maturity less or equal to the horizon *T*. For each instants $u \le T$, we define an adapted process $(B(t, u))_{0 \le t \le u}$, satisfying $B(u. u) = 1$ giving the price of the zerocoupon bond with maturity *u* as a function of time.

By characterized the absence of arbitrage opportunities by the existence of an equivalent probability under which discounted asset prices are martingales. The extension of this result to the continuous-time models is rather technical (cf. Harrison and Kreps, 1979; Artzner and Delbaen, 1989; Stricker, 1990; Delbaen and Schachermayer, 1994), but we are able to check that such a probability exists in the Black-Scholes model. In the light of this examples, the starting point of the modeling will be passed upon the following hypothesis:

(**H**) There is a probability P^* equivalent to *P*, under which, for all real valued $u \in [0, T]$, the process $(\tilde{B}(t, u))_{0 \le t \le u}$ defined by

$$\tilde{B}(t, u) = e^{-\int_0^t r(s)ds} B(t, u)$$

is a martingale.

This hypothesis has some interesting consequences. Indeed, the martingale property under P^* leads to, using the equality $B(u, u) = 1$,

$$\tilde{B}(t, u) = E^*\left(\tilde{B}(u, u) \mid \mathcal{F}_t\right) = E^*\left(e^{-\int_0^u r(s)ds} \mid \mathcal{F}_t\right)$$

and, eliminating the discounting,

$$B(t, u) = E^*\left(e^{-\int_0^u r(s)ds} ds \mid \mathcal{F}_t\right)$$

Here we study the pricing problem for an underlying asset coupon bond which is governed by the following stochastic differential equation

$$\frac{dB(t, u)}{B(t, u)} = \left(r(t) - \sigma_t^u \lambda(t)\right) dt + \sigma_t^u dW_t \quad \text{...(1.1)}$$

The following proposition gives an economic interpretation of the process $(\lambda(t))$

Proposition 1.1: For each maturity *u*, there is an adapted process $(\sigma_t^u)_{0 \le t \le u}$ such that, on [0, *u*]

$$\frac{dB(t, u)}{B(t, u)} = \left(r(t) - \sigma_t^u \lambda(t)\right) dt + \sigma_t^u dW_t$$

Proof: Since the process $(\tilde{B}(t, u))_{0 \le t \le u}$ is a martingale under P^*, $(\tilde{B}(t, u) L_t)_{0 \le t \le u}$ is a martingale under *P*. Moreover, we have: $\tilde{B}(t, u) L_t > 0$ *a.s* for all $t \in [0, u]$. Then, there exists an adapted process $(\theta_t^u)_{0 \le t \le u}$ such that $\int_0^u (\theta_t^u)^2 dt < \infty$ and

$$\tilde{B}(t, u) L_t = \tilde{B}(0, u) e^{-\int_0^t \theta_s^u dW_s - \frac{1}{2}\int_0^t (\theta_s^u)^2 ds}$$

Hence, using the explicit expression of L_t and getting rid of the discounting factor

$$B(t, u) = B(0, u) e^{\int_0^t r(s)ds + \int_0^t (\theta_s^u - \lambda(s))dW_s - \frac{1}{2}\int_0^t \left((\theta_s^u)^2 - \lambda(s)\right)^2 ds}$$

Applying the $It\hat{o}$ formula with the exponential function, we get

$$\frac{dB(t, u)}{B(t, u)} = r(t)dt + (\theta_t^u - \lambda(t))dW_t - \frac{1}{2}\left((\theta_t^u)^2 - \lambda(t)^2\right)dt + \frac{1}{2}(\theta_t^u - \lambda(t))^2 dt$$

$$= \left(r(t) + \lambda(t)^2 - \theta_t^u \lambda(t)\right)dt + (\theta_t^u - \lambda(t))dW_t$$

Which gives the equality (1.1) with $\sigma_t^u = \theta_t^u - \lambda(t)$.

dW_t which makes the bond riskier. Furthermore, the term $r(t) - \sigma_t^u \lambda(t)$ corresponds intuitively to the average yield (i.e., in expectation) of the bond at time *t* (because increment of Brownian motion have zero expectation) and the term $-\sigma_t^u \lambda(t)$ is the difference between the average yield of the bond and the riskless rate, hence the interpretation of $-\lambda(t)$ as a 'risk premium'. Under probability *P**, the process $(\tilde{W}_t)$ defined by $\tilde{W}_t = W_t - \int_0^t \lambda(s)ds$ is a standard Brownian motion (Girsanov theorem), and we have

$$\frac{dB(t, u)}{B(t, u)} = r(t)dt + \sigma_t^u d\tilde{W}_t \qquad ...(1.2)$$

For this reason, the probability *P** is often called the 'risk neutral' probability.

1.2. Bond Options

To make things clearer, let us first consider a European option with maturity θ on the zero-coupon bond with maturity equal to the horizon *T*. If it is a call with strike price *K*, the value of the option at time θ is obviously $(B(\theta, T)-K)^+$ and it seems reasonable to hedge this call with a portfolio of riskless asset and zero-coupon bond with maturity *T*. A strategy is then defined by an adapted process $\left((H_t^0, H_t)\right)_{0\le t\le T}$ with values in R^2, H_t^0 representing the quantity of riskless asset and H_t the number of bonds with maturity *T* held in the portfolio at time *t*. The values of the portfolio at time *t* are given by

$$V_t = H_t^0 B_t + H_t B(t, T) = H_t^0 e^{\int_0^t r(s)ds} + H_t B(t, T)$$

and the self-financing condition is written as

$$dV_t = H_t^0 dB_t + H_t dB(t, T)$$

Taking into account proposition 1.1, we impose the following integrability conditions: $\int_0^T \left|H_t^0 r(t)\right| dt < \infty$ and $\int_0^T (H_t \sigma_t^u)^2 dt < \infty$ *a.s.* We will define admissible strategies in the following manner:

Definition 1.3: A strategy $\varphi = \left((H_t^0, H_t)\right)_{0\le t\le T}$ is admissible if it is self-financing and if the discounted value $\tilde{V}_t = H_t^0 + H_t \tilde{B}(t, T)$ of the corresponding portfolio is, for all *t*, non-negative and if $\sup_{t\in[0, T]} \tilde{V}_t$ is square-integrable under *P**.

The following proposition shows that under some assumptions, it is possible to hedge all European options with maturity $\theta < T$.

Proposition 1.4: We assume $\sup_{0\le t\le T}|r(t)|<\infty$ *a.s* and $\sigma_t^T \neq 0$ *a.s* for all $t\in[0,\theta]$. Let $\theta<T$ and let h be an $\mathcal{F}_\theta$-measurable random variable such that $he^{-\int_0^\theta r(s)ds}$ is square-integrable under P^*. Then there exists an admissible strategy whose value at time θ is equal to h. The value at time $t\le\theta$ of such a strategy is given by

$$V_t = E^*\left(e^{-\int_0^\theta r(s)ds} h \middle| \mathcal{F}_t\right)$$

Proof: We first observe that if $\tilde{V}_t$ is the value at time t of an admissible strategy $\left((H_t^0, H_t)\right)_{0\le t\le T}$, we obtain, using the self-financing condition, the integration by parts formula and remark 1.2 (cf. Equation 1.2)

$$d\tilde{V}_t = H_t d\tilde{B}(t,T) = H_t\tilde{B}(t,T)\sigma_t^T d\tilde{W}_t$$

We deduce, bearing in mind that $\sup_{t\in[0,T]}\tilde{V}_t$ is square-integrable under P^*, that $(\tilde{V}_t)$ is a martingale under P^*.

Thus, we have

$$\forall t\le\theta\ \tilde{V}_t = E^*\left(\tilde{V}_\theta \middle| \mathcal{F}_t\right)$$

and if we impose the condition $V_\theta = h$, we get

$$V_t = e^{\int_0^t r(s)ds} E^*\left(e^{-\int_0^\theta r(s)ds} h \middle| \mathcal{F}_t\right)$$

To complete the proof, it is sufficient to find an admissible strategy having the same value at any time. To do so, one proves that there exists a process $(j_t)_{0\le t\le\theta}$ such that $\int_0^\theta j_t^2 dt<\infty$, *a.s.* and

$$he^{-\int_0^\theta r(s)ds} = E^*\left(he^{-\int_0^\theta r(s)ds}\right) + \int_0^\theta j_s d\tilde{W}_s$$

Note that this property is not trivial consequence of the theorem of representation of martingales because we do not know where $he^{-\int_0^\theta r(s)ds}$ is in the σ-algebra generated by the $\tilde{W}_t$, $t\le\theta$, (we only know it is in the σ-algebra $\mathcal{F}_\theta$ which can be bigger). Once this property is proved, it is sufficient to set

$$H_t = \frac{j_t}{\tilde{B}(t,T)\sigma_t^T} \quad \text{and} \quad H_t^0 = E^*\left(he^{-\int_0^\theta r(s)ds} \middle| \mathcal{F}_t\right) - \frac{j_t}{\sigma_t^T}$$

for $t\le\theta$. We check easily that $\left((H_t^0, H_t)\right)_{0\le t\le\theta}$ defines an admissible strategy (the hypothesis $\sup_{0\le t\le T}|r(t)|<\infty$ *a.s* guarantee that the condition $\int_0^\theta |r(s)H_s^0|ds<\infty$ holds) whose value at time θ is indeed equal to h. □

Remark 1.5: We have not investigated the uniqueness of the probability P^* and it is not clear that the risk process $(\lambda(t))$ is defined without ambiguity. Actually, it can be shown (cf. Artzner and Delbaen, 1989) that P^* is the unique probability equivalent to P under which $\left(\tilde{B}(t,T)\right)_{0\le t\le T}$ is a martingale if and only if the process (σ_t^T) satisfies $\sigma_t^T\neq 0$, $dtdP$ almost everywhere. This condition, slightly weaker than the hypothesis of proposition 1.4 is exactly what is needed to hedge options with bonds of maturity T, which is not surprising when one keeps in mind the characterization of a complete markets.

1.3. The Model

The main drawback of the Vasicek model and the Cox-Ingersoll-Ross model lies in the fact that prices are explicit functions of the instantaneous interest rate so that these models are unable to take the whole yield curve observed on the market into account in the price structure.

Some authors have restored to a two-dimensional analysis to improve the models in terms of discrepancies between short and long rates, cf. Brennan and Schwartz (1979), Schaefer and Schwartz (1984) and Courtadon (1982).

These more complex models do not lead to explicit formulae and require the solution of partial differential equations. More recently, Ho and Lee (1986) have proposed a discrete-time model describing the behavior of the whole yield curve. The continuous-time model we will present now is based on the same idea.

First of all, we define the forward interest rates $f(t, s)$, for $t \le s$, characterized by the following equality:

$$B(t, u) = \exp\left(-\int_t^u f(t, s)\,ds\right) \qquad ...(1.3)$$

for any maturity u. So $f(t, s)$ represents the instantaneous interest rate at times s as 'anticipated' by the market at time t. For each u the process $\left(f(t, u)\right)_{0\le t\le u}$ must then be an adapted process and it is natural to set $f(t, t) = r(t)$. Moreover, we constrain the map $(t, s) \to f(t, s)$, defined for $t \le s$, to be continuous. Then the next step of the modeling consists in assuming that, for each maturity u the process $\left(f(t, u)\right)_{0\le t\le u}$ satisfies an equation of the following form:

$$f(t, u) = f(0, u) + \int_0^t \alpha(v, u)\,dv + \int_0^t \sigma\left(f(v, u)\,dW_v\right) \qquad ...(1.4)$$

The process $\left(\alpha(t, u)\right)_{0\le t\le u}$ being adapted, the map $(t, u) \to \alpha(t, u)$ being continuous and σ being a continuous map from R into R (σ could depend on time as well, cf. Morton (1989)).

Then we have to make sure that this model is compatible with the hypothesis (**H**). This gives some conditions on the coefficients α and σ of the model. To find them, we derive the differential $dB(t, u)/B(t, u)$ and we compare it to Equation (1.1). Let us set $X_t = -\int_t^u f(t, s)\,ds$. We have $B(t, u) = e^{X_t}$ and from Equation (1.4)

$$\begin{aligned} X_t &= \int_t^u \left(-f(s, s) + f(s, s) - f(t, s)\right)ds \\ &= -\int_t^u f(s, s)\,ds + \int_t^u \left(\int_t^s \alpha(v, s)\,dv\right)ds + \int_t^u \left(\int_t^s \sigma\left(f(v, s)\right)dW_v\right)ds \\ &= -\int_t^u f(s, s)\,ds + \int_t^u \left(\int_v^s \alpha(v, s)\,ds\right)dv + \int_t^u \left(\int_v^s \sigma\left(f(v, s)\right)ds\right)dW_v \\ &= X_0 + \int_0^t f(s, s)\,ds - \int_0^t \left(\int_v^u \alpha(v, s)\,ds\right)dv - \int_0^t \left(\int_v^u \sigma\left(f(v, s)\right)ds\right)dW_v \end{aligned} \qquad ...(1.5)$$

The fact that the integrals commute in Equation (1.5) is justified in the Proposition 1.6 we then have

$$dX_t = \left(f(t, t) - \int_t^u \alpha(t, s)\,ds\right)dt - \left(\int_t^u \sigma\left(f(t, s)\right)ds\right)dW_t$$

and by the $It\hat{o}$ formula

$$\begin{aligned} \frac{dB(t, u)}{B(t, u)} &= dX_t + \frac{1}{2}d\langle X, X\rangle_t \\ &= \left(f(t, t) - \left(\int_t^u \alpha(t, s)\,ds\right) + \frac{1}{2}\left(\int_t^u \sigma\left(f(t, s)\right)ds\right)^2\right)dt - \left(\int_t^u \sigma\left(f(t, s)\right)ds\right)dW_t \end{aligned}$$

If the hypothesis (**H**) holds, we must have, from proposition (1.1) and equality $f(t, t) = r(t)$

$$\sigma_t^u \lambda(t) = \left(\int_t^u \alpha(t, s)\,ds\right) - \frac{1}{2}\left(\int_t^u \sigma(f(t, s))\,ds\right)^2$$

with $\sigma_t^u = -\left(\int_t^u \sigma(f(t, s))\,ds\right)$. Whence

$$\int_t^u \alpha(t, s)\,ds = \frac{1}{2}\left(\int_t^u \sigma(f(t, s))\,ds\right)^2 - \lambda(t)\int_t^u \sigma(f(t, s))\,ds$$

and, differentiating with respect to u

$$\alpha(t, u) = \sigma(f(t, u))\left(\int_t^u \sigma(f(t, s))\,ds - \lambda(t)\right)$$

Equation (1.4) if written in differential form, becomes

$$df(t, u) = \sigma(f(t, u))\left(\int_t^u \sigma(f(t, s)\,ds)\,dt + \sigma(f(t, u))\,d\tilde{W}_t\right) \quad ...(1.6)$$

Proposition 1.6: Let $(\Omega, F, (\mathcal{F}_t)_{0\le t\le T}, P)$ be a filtered space and let $(W_t)_{0\le t\le T}$ be a standard Brownian motion with respect to ($\mathcal{F}_t$). We consider process with two indices $(H(t, s))_{0\le t,\, s\le T}$ satisfying the following properties: for any W, the map $(t, s) \to H(t, s)(\omega)$ is continuous and for any $s \in [0, T]$ the process $(H(t, s))_{0\le t\le T}$ is adapted. We would like to justify the equality

$$\int_0^T\left(\int_0^T H(t, s)\,dW_t\right)ds = \int_0^T\left(\int_0^T H(t, s)\,ds\right)dW_t$$

For simplicity we assume that $\int_0^T E\left(\int_0^T H^2(t, s)\,dt\right)ds < \infty$ (which is sufficient to justify Equation (1.5))

Note that the boundedness condition on σ is essential case, for $\sigma(x) = x$, there is no solution.

1.4. Absence of Arbitrage

In the present setting, a continuum of bonds with different maturities is available for trade. We will assume, however, that any particular portfolio involves investments in an arbitrary, but finite, number of bonds. An alternative approach, in which infinite portfolios are also allowed.

For any collection of maturities $0 < T_1 < T_2 < ...... < T_k = T^*$, we write $\mathcal{T}$ to denote the vector $(T_1, ..., T_k)$; similarly, $B(., \mathcal{T})$ stands for the $\mathbb{R}^k$ -valued process $(B(t, T_1), ..., B(t, T_k))$. We find it convenient to extend the $\mathbb{R}^k$ -valued process $B(., T)$ over the time interval $[0, T^*]$by setting $B(t, T) = 0$ for any $t \in (T, T^*]$ and any maturity $0 < T < T^*$.

By a bond trading strategy, we mean a pair $(\phi, \mathcal{T})$, where ϕ is a predictable $\mathbb{R}^k$ -valued stochastic process that satisfies $\phi_t^i = 0$ for every $t \in (T_i, T^*]$ and any $i = 1, 2, ..., k$. A bond trading strategy $(\phi, \mathcal{T})$ is said to be self-financing if the wealth process $V(\phi)$, which equals

$$V_t(\phi) \overset{def}{=} \phi_t \cdot B(t, T) = \sum_{i=1}^{k} \phi_t^i B(t, T_i)$$

Satisfies

$$V_t(\phi) = V_0(\phi) + \int_0^t \phi_u \cdot dB(u, T) = V_0(\phi) + \sum_{i=1}^{k}\int_0^t \phi_u^i\,dB(u, T_i)$$

for every $t \in [0, T^*]$. To ensure the arbitrage-free properties of the bond market model, we need to examine the existence of a martingale measure for a suitable choice of a numeraire; in the present set-up, we can take either the bond price $B(t, T^*)$or the savings account B.

Assume for simplicity, that the coefficient σ in Equation (1.4) is bounded. We are looking for a condition ensuring the absence of arbitrage opportunities across all bonds of different maturities.

In order to formulate out arbitrage opportunities between all bonds with different maturities, it suffices to assume that a martingale measure $\hat{P}$ can be chosen simultaneously for all maturities.

The following condition is thus sufficient for the absence of arbitrage between all bonds. As usual, we restrict our attention to the class of admissible trading strategies.

Condition (M.1): There exist an adapted $\mathbb{R}^d$ -valued process h such that

$$E_p\left\{\varepsilon_{T^*}\int_0 h_u dW_u\right\} = 1$$

and for every $T \le T^*$

$$\int_T^{T^*} \alpha(t, u)\,du + \frac{1}{2}\left|\int_T^{T^*} \sigma(t, u)\,du\right|^2 + h_t \cdot \int_T^{T^*} \sigma(t, u)\,du = 0$$

By taking the partial derivative with respect to T, we obtain

$$\alpha(t, T) + \sigma(t, T)\cdot\left(h_t + \int_T^{T^*} \sigma(t, u)\,du\right) = 0 \qquad \text{...(1.7)}$$

for every $0 \le t \le T \le T^*$. For any process h of condition (**M.1**), the probability measure $\hat{P}$. Given by $\frac{d\tilde{P}}{dP} = \varepsilon_{T^*}\left(\int_0^{\cdot} h_u \cdot dW_u\right), p - a.s.$ Will later be interpreted as the forward martingale measure for the date T^*. Assume now, in addition, that one may invest also in the saving accounts given by $B_t = \exp\left(\int_0^t f(u, u)\,du\right), \forall t \in [0, T^*]$. The relative bond price $Z^*(t, T) = {B(t, T)}/{B_t}$ satisfies under P

$$dZ^*(t, T) = -Z^*(t, T)\left(\left(\alpha^*(t, T) - \frac{1}{2}\left|\sigma^*(t, T)\right|^2\right)dt + \sigma^*(t, T)\,dW_t\right)$$

where

$$\alpha^*(t, T) = \int_t^T \alpha(t, u)\,du \text{ and } \sigma^*(t, T) = \int_t^T \sigma(t, u)\,du$$

The following no-arbitrage condition excludes arbitrage not only across all bonds, but also between bonds and the savings account.

Condition (M.2): There exists an adapted $\mathbb{R}^d$ -valued process λ such that

$$E_P\left\{\varepsilon_{T^*}\int_0^{\cdot} \lambda_u \cdot dW_u\right\} = 1$$

and, for any maturity $T \le T^*$, we have

$$\alpha^*(t, T) = \frac{1}{2}\left|\sigma^*(t, T)\right|^2 - \sigma^*(t, T)\cdot\lambda_t$$

Differentiation of the last equality with respect to T yields the equality

$$\alpha(t, T) = \sigma(t, T) \cdot \left(\sigma^*(t, T) - \lambda_t\right),\ \forall t \in [0, T] \qquad ...(1.8)$$

Which holds for any $T \leq T^*$. A probability measure P^* satisfying

$$\frac{dP^*}{dP} = \varepsilon_{T^*}\left[\int_0^{\cdot} \lambda_u \cdot dW_u\right] \ P - a.s$$

for some process λ satisfying (**M.2**), can be seen as a spot martingale measure for the HJM model; in this context, the process λ is associated with the risk premium.

Define a P^*-Brownian motion W^* by setting

$$W_t^* = W_t - \int_0^t \lambda_u du,\ \forall t \in [0, T]$$

The next result, whose proof is straightforward, deals with the dynamics of bond price and interest rates under the spot martingale measure P^*.

Corollary 1.7 For any fixed maturity $T \leq T^*$, the dynamics of the bond price $B(t, T)$ under the spot martingale measure P^* are

$$dB(t, T) = B(t, T)\left(r_t dt - \sigma^*(t, T) dW_t^*\right) \qquad ...(1.9)$$

and the forward rate $f(t, T)$ satisfies

$$df(t, T) = \sigma(t, T) \cdot \sigma^*(t, T) dt + \sigma(t, T) \cdot dW_t^* \qquad ...(1.10)$$

Finally, the short-term interest rate $r_t = f(t, t)$ is given by the expression

$$r_t = f(0, t) + \int_0^t \sigma(u, t) \cdot \sigma^*(u, t) du + \int_0^t \sigma(u, t) \cdot dW_u^* \qquad ...(1.11)$$

It follows from (1.11) that the expectation of the future short-term rate under the spot martingale measure P^* does not equal the current value $f(0, T)$ of the instantaneous forward rate; that is, $f(0, T) \neq E_{P^*}(r_T)$, in general. $f(0, T)$ equals the expectation of r_T under the forward martingale measure for the date T. In view of Equation (1.9) the relative bond price $Z^*(t, T) = {B(t, T)}/{B_t}$ satisfies

$$dZ^*(t, T) = -Z^*(t, T)\sigma^*(t, T) \cdot dW_t^* \qquad ...(1.12)$$

and thus

$$Z^*(t, T) = B(0, T)\exp\left(-\int_0^t \sigma^*(u, T) \cdot dW_u^*\right) - \frac{1}{2}\int_0^t \left|\sigma^*(u, T)\right|^2 du$$

Or equivalently

$$\ln B(t, T) = \ln B(0, T) + \int_0^t \left(r_u - \frac{1}{2}\left|\sigma^*(u, T)\right|^2\right) du - \int_0^t \sigma^*(u, T) \cdot dW_t^*$$

Remarks: It is possible to go the other way around; that is, to assume that the dynamics of $B(t, T)$ under the martingale measure P^* are

$$dB(t, T) = B(t, T)\left(r_t dt + b(t, T) \cdot dW_t^*\right)$$

where r is the short-term rate process, and the volatility coefficient b is differentiable with respect to maturity T. Then clearly

$$\ln B(t, T) = \ln B(0, T) + \int_0^t \left(r_u - \frac{1}{2}|b(u, T)|^2 \right) du + \int_0^t b(u, T) \cdot dW_u^*$$

By differentiating this equality with respect to T. We find the dynamics of the forward rate under P^*, namely

$$f(t, T) = f(0, T) + \frac{1}{2}\int_0^t \frac{\partial k(u, T)}{\partial T} du + \int_0^t \frac{\partial b(u, T)}{\partial T} \cdot dW_u^* \qquad ...(1.13)$$

where we write $k(t, T) = |b(t, T)|^2$. Note that we have interchanged the differentiation with the stochastic integration. This is essentially equivalent to an application of the stochastic Fubini theorem.

It is not hard to check that, under mild technical assumptions, the noarbitrage conditions (**M.1**) and (**M.2**) are equivalent.

Lemma 1.8: Conditions (**M.1**) and (**M.2**) are equivalent.

Proof: Let us sketch the proof. Suppose first that condition (**M.1**) holds.

Let us define λ by setting

$$\lambda_t \overset{def}{=} h_t - b(t, T^*) = h_t + \int_t^{T^*} \sigma(t, u) du \quad \forall t \in [0, T^*]$$

and check that for such a process λ condition (1.8) is satisfied. Indeed for the right-hand side of Equation (1.8), we obtain

$$\sigma(t, T) \cdot (\sigma^*(t, T) - \lambda_t) = -\sigma(t, T) \cdot \left(h_t + \int_T^{T^*} \sigma(t, u) du \right) = \alpha(t, T)$$

where the last equality follows from Equation (1.7). Conversely, given a process λ that satisfies Equation (1.8) we define the process h by setting

$$h_t \overset{def}{=} \lambda_t + b(t, T^*) \quad \forall t \in [0, T^*]$$

Then h satisfies Equation (1.7) since

$$\sigma(t, T) \cdot \left(h_t + \int_T^{T^*} \sigma(t, u) du \right) = \sigma(t, T) \cdot (\lambda_t - \sigma^*(t, T)) = -\alpha(t, T)$$

where the last equality follows from Equation (1.8). □

We assume from now on that the following assumption is satisfied. (HJM.1) No-arbitrage condition (**M.1**) (or equivalently (**M.2**)) is satisfied. It is not essential to assume that the martingale measure for the bond market is unique, so long as we are not concerned with the completeness of the model. Recall that if a market model is arbitrage, free, any attainable claim admits a unique arbitrage price anyway. It is uniquely determined by the replicating strategy, whether a market model is complete or not. For instance, in a Gaussian HJM setting, several claims-such as options on a bond or a stock-are attainable, with replicating strategies given by closed form expressions. One may argue that from the practical viewpoint, the possibility of explicit replication of most typical claims is a more important feature of a model than its theoretical completeness.

Remark 1.9: As was mentioned before, it seems plausible that σ might depend explicitly on the level of the forward rate. This corresponds to the following forward interest rate dynamics.

$$df(t, T) = \alpha(t, T, f(t, T))dt + \sigma(t, T, f(t, T)) \cdot dW_t \qquad ...(1.14)$$

where the deterministic function $\sigma : C \times \mathbb{R} \to \mathbb{R}$ is sufficiently regular, so that SDE (1.14) (with the drift coefficient α given by (1.15)) admits a unique strong solution for any fixed maturity T.

Recall that the drift coefficient α satisfies under the martingale measure P^*

$$\alpha\big(t, T, f(t, T)\big) = \sigma\big(t, T, f(t, T)\big) \cdot \int_t^T \sigma\big(t, u, f(t, u)\big) du \quad ...(1.15)$$

Consequently, under the martingale measure P^* SDE (1.14) may be rewritten in the following form

$$df(t, T) = \sigma\big(t, T, f(t, T)\big) \cdot \Big(\int_t^T \sigma\big(t, u, f(t, u) dudt + dW_t^*\big) \Big) \quad ...(1.16)$$

Let us assume, for instance, that the coefficient σ satisfies

$$\sigma\big(t, T, f(t, T)\big) = \gamma f(t, T) \qquad \forall t \in [0, T],$$

where γ is a fixed vector in $\mathbb{R}^d$. Under such an assumption (1.16) becomes

$$f(t, T) = f(0, T) + \int_0^t |\gamma|^2 f(u, T) \int_u^T f(u, v) dv du + \int_0^t f(u, T) \gamma \cdot dW_u^*$$

for any maturity date $T \in [0, T^*]$. Since the drift coefficient grows rapidly as a function of the forward rate, the last stochastic differential equation does not admit a global solution; indeed, it can be shown that its local solution explodes in a finite time. For more details on this important issue, we referred to Morton (1989) and Miltersen (1994).

1.5. Short-Term Interest Rate

It is interesting to note that under some regularity assumptions, the short-term interest rate process specified by the HJM model of the instantaneous forward rates follows a continuous semi martingale, or even a diffusion process (if the regularity of α and σ with respect to maturity T is not assumed, the behavior of the short-term rate is rather difficult to study). The following proposition can be seen as the first step in this direction. Further result and typical examples are given in the next section.

Proposition 1.10: Suppose that the coefficients $\alpha(t, T)$ and $\sigma(t, T)$ and the initial forward curve $f(0, T)$ are differentiable with respect to maturity T, with bounded partial derivatives $\alpha_T(t, T)$, $\sigma_T(t, T)$ and $f_T(0, T)$. Then the short interest rate r follows a continuous semi martingale under P. More specifically, for any $t \in [0, T^*]$ we have

$$r_t = r_0 + \int_0^t \xi_u du + \int_0^t \sigma(u, u) \cdot dW_u \quad ...(1.17)$$

where ξ stands for the following process

$$\xi_t = \alpha(t, t) + f_T(0, t) + \int_0^t \alpha_T(u, t) du + \int_0^t \sigma_T(u, t) \cdot dW_u$$

Proof: Observe first that r satisfies

$$r_t = f(t, t) = f(0, t) + \int_0^t \alpha(u, t) du + \int_0^t \sigma(u, t) \cdot dW_u$$

Apply the stochastic Fubini theorem to the $It\hat{o}$ integral, we obtain

$$\int_0^t \sigma(u, t) \cdot dW_u = \int_0^t \sigma(u, u) \cdot dW_u + \int_0^t \big(\sigma(u, t) - \sigma(u, u)\big) \cdot dW_u$$

$$= \int_0^t \sigma(u, u) \cdot dW_u + \int_0^t \int_u^t \sigma_T(u, s) ds \cdot dW_u$$

$$= \int_0^t \sigma(u, u) \cdot dW_u + \int_0^t \int_0^s \sigma_T(u, s) \cdot dW_u ds$$

Furthermore

$$\int_0^t \alpha(u, t)\, du = \int_0^t \alpha(u, u)\, du + \int_0^t \int_0^s \alpha_T(u, s)\, duds$$

and finally

$$f(0, t) = r_0 + \int_0^t f_T(0, u)\, du$$

Combining these formulas, we obtain (1.17)

1.6. Gaussian HJM Model

In this section, we assume that the volatility σ of the forward rate is deterministic; such a case will be referred to as the Gaussian HJM model. This terminology refers to the fact that the forward rate $f(t, T)$ and the spot rate r_t have Gaussian probability distributions under the martingale measure P^* (cf. Equations (1.10)-(1.11)). Our aim is to show that the arbitrage price of any attainable interest rate-sensitive claim can be evaluated by each of the following procedures.

1. We started with arbitrary dynamics of the forward rate such that condition (**M.1**) (or (**M.2**)) is satisfied. We then find a martingale measure P^*, and apply the risk-neutral valuation formula.
2. We assume instead that the underlying probability measure P is actually the spot martingale measure. In other words, we assume that condition (**M.2**) (respectively, condition (**M.1**)) is satisfied, with the process λ (respectively, h) equal to zero.

Since both procedures give the same valuation results, we conclude that the specification of the risk premium is not relevant in the context of arbitrage valuation of interest rate-sensitive derivatives in the Gaussian HJM framework. Put differently, when the coefficient σ is deterministic, we can assume, without loss of generality that $\alpha(t, T) = \sigma(t, T)\sigma^*(t, T)$, observe that $f(t, t) = r_t$. To state a result that formally justifies the considerations above, we need to introduce some additional notation. Let the function α_0 be given by Equation (1.8), with $\lambda = 0$, i.e.,

$$\alpha_0(t, T) = \sigma(t, T)\sigma^*(t, T), \qquad \forall t \in [0, T] \qquad ...(1.18)$$

So that

$$\alpha_0^*(t, T) = \int_t^T \alpha_0(u, T)\, du = \frac{1}{2}\left|\sigma^*(t, T)\right|^2$$

Finally, we denote by $B_0(t, T)$ the bond price specified by the equality

$$B_0(t, T) = \exp\left(-\int_t^T f_0(t, u)\, du\right)$$

where the dynamics under P of the instantaneous forward rate $f_0(t, T)$ are

$$f_0(t, T) = f(0, T) + \int_0^t \alpha_0(u, T)\, du + \int_0^t \sigma(u, T) \cdot dW_u$$

Let us put

$$Z^*(t, \mathcal{T}) = B(t, \mathcal{T}) / B_t, \; Z_0^*(t, T) = B_0(t, T) / B_t$$

where ($\mathcal{T} = T_1, ..., T_k$) is any finite collection of maturity dates.

Proposition 1.11: Suppose that the coefficient σ is deterministic. Then for any choice $\mathcal{T}$ of maturity dates and of a spot martingale measure P^*, the probability distribution of the process $Z^*(t, T), t \in [0, T^*]$, under the martingale measure P^* coincides with the probability distribution of the process $Z_0^*(t, \mathcal{T}), t \in [0, T^*]$, under P.

Proof: The assertion follows easily by Girsanov's theorem. Indeed, for any fixed $0 < T \le T^*$ the dynamics of $Z^*(t, T)$ under a spot martingale measure $P^* = P^\lambda$ are

$$dZ^*(t, T) = -Z^*(t, T)\sigma^*(t, T)\cdot dW_t^\lambda \quad ...(1.19)$$

where W^λ follows a standard Brownian motion under P^λ. On the other hand, under P we have

$$dZ_0^*(t, T) = -Z_0^*(t, T)\sigma^*(t, T)\cdot dW_t \quad ...(1.20)$$

Moreover, for every $0 < T \le T^*$

$$Z^*(0, T) = B(0, T) = e^{-\int_0^T f(0, u)du} = B_0(0, T) = Z_0^*(0, T)$$

Since σ is deterministic, then the assertion follows easily from Equations (1.19)-(1.20).

To show the independence of the arbitrage pricing of the market prices for risk in a slightly more general setting, it is convenient to make use of the savings account. Under any martingale measure P^* we have

$$df(t, T) = \sigma(t, T, f(t, T))\cdot\left(\int_t^T \sigma(t, u, f(t, u))dudt + dW_t^*\right) \quad ...(1.21)$$

We assume that the coefficient σ satisfies regularity assumptions, so that the SDE (1.21) admits a unique strong global solution. Since

$$B(t, T) = \exp\left(-\int_t^T f(t, u)du\right), \qquad \forall t \in [0, T]$$

and

$$B_t = \exp\left(\int_0^t f(u, u)du\right), \qquad \forall t \in [0, T^*]$$

it is clear that the joint probability distribution of processes $B(., T)$ and B is uniquely determined under P^*, and thus the arbitrage price $\pi_t(X)$ of any attainable European claim X depending on short-term rate and bond prices (or on the forward rates), which equals

$$\pi_t(X) = B_t E_{P^*}\left(XB_T^{-1} \middle| \mathcal{F}_t\right)$$

is obviously independent of the market prices for risk. This does not mean that the market prices for risk are neglected altogether in the HJM approach. They are still present in the specification of the actual bond price fluctuations.

However, in contrast to traditional models of the short-term rate, in which the dynamics of the short short-term rate and bond processes are not jointly specified, the HJM methodology assumes a simultaneous influence of market prices for risk on the dynamics of the short-term rate and all bond prices. Consequently, in this case the market prices for risk drop out altogether from arbitrage values of interest rate-sensitive derivatives.

1.7. Markovian Case

The HJM approach typically produces models in which the short-term rate has path-dependent features. A discrete-time approximation within the HJM framework is therefore usually less efficient than in the path-independent case, since the number of operations rises exponentially with the number of operations rises exponentially with the number Jamshidian (1991a), Carverhill (1994), Jeffrey (1995), Ritchken and Sankarasubramanian (1995), implied short-term interest rate r has the Markov property. Assume that the bond price volatility $b(t, T)$ is a deterministic function that is twice continuously $\sigma(t, T)$ differentiable with respect to maturity date T (equivalently, that is continuously differentiable with respect to T), from (1.11) we find that r satisfies

$$r_t = f(0, t) + \int_0^t \sigma(u, t)\cdot\sigma^*(u, t)du + \int_0^t \sigma(u, t)\cdot dW_u^* \quad ...(1.22)$$

To show that the short-term rate r has the Markov property with respect to the filtration $\mathcal{F}^{W^*}$ under P^* it is enough to show that for any bounded Borel-measurable function $h: \mathbb{R} \to \mathbb{R}$, and any $t \le S \le T^*$, we have

$$E_{P^*}\left(h(r_s)\middle|\mathcal{F}_t^{w}\right) = E_{P^*}\left(h(r_s)\middle|r_t\right),$$

where on the right-hand side we have the conditional expectation with respect to the σ-field $\sigma(r_t)$; that is, the σ-field generated by the random variable r_t. For the sake of brevity, we say that r is Markovian if it has the Markov property with respect to the filtration $\mathcal{F}_t^{W^*}$ under P^*.

Proposition 1.12: Suppose that the short-term rate r is Markovian. Assume, in addition, that for any maturity $T \le T^*$ we have $\sigma(t, T) \neq 0$ for every $t \in [0, T]$. Then there exist functions $g:[0, T^*] \to \mathbb{R}$ R and $h:[0, T^*] \to R$ such that

$$\sigma(t, T) = h(t)g(T) \qquad \forall t \le T \le T^* \qquad \text{...(1.23)}$$

Moreover, we have

$$b^i(t, T) = \frac{\sigma^i(t, T^*)}{\sigma^i(s, T^*)} \cdot \left(b^i(s, T) - b^i(s, t)\right), \qquad \forall t \le s \le T$$

for every $T \in [0, T^*]$ and every i = 1, ..., d. If the volatility $\sigma(t, T)$ is also time-homogeneous, that is, $\sigma(t+s, T+s) = \sigma(t, T)$ for all $s \ge 0$. Then there exist constants α^i and γ such that for every i = 1, ..., d we have

$$b^i(t, T) = \alpha^i\left(1 - e^{-\gamma(T-t)}\right), \qquad \forall t \le T \le T^* \qquad \text{...(1.24)}$$

Proof: In view of Equation (1.22), it is clear that the short-term rate r is Markovian

if and only if the process $D_t = \int_0^t \sigma(u, t) \cdot dW_u^*$ is Markovian-that is, if

$$E_{P^*}\left(h(D_s)\middle|\mathcal{F}_t^{D}\right) = E_{P^*}\left(h(D_s)\middle|D_t\right), \qquad \forall t \le s \le T^* \qquad \text{...(1.25)}$$

For any bounded Borel. Measurable function $h: \mathbb{R} \to \mathbb{R}$. Observe that

$$D_s = D_t + \int_0^s \sigma(u, s) \cdot dW_u^* - \int_0^t \sigma(u, t) \cdot dW_u^*$$

and thus also

$$D_s = D_t + \int_0^s \sigma(u, s) \cdot dW_u^* + \int_0^t \left(\sigma(u, s) - \sigma(u, t)\right) \cdot dW_u^*$$

In view of the last formula, Equation (1.25) is valid if and only if, given the random variable D_t, the integral

$$I(t, s) = \int_0^t \left(\sigma(u, s) - \sigma(u, t)\right) \cdot dW_u^* = \int_0^t \sigma(u, s) \cdot dW_u^* - D_t$$

depends only on the increments of the Brownian motion W^*on $[t, s]$. But $I(t, s)$ is for obvious reasons, independent of these increments, therefore we conclude that the integral $J(t, s) = \int_0^t \sigma(u, s) \cdot dW_u^*$ is completely determined when D_t is given. Since the joint distribution of $(J(t, s)D_t)$ is Gaussian, this is means that the correlation coefficient satisfies, $|Corr(j(t, s)D_t)| = 1$, or more explicitly

$$\left(E_{P^*}\left(J(t, s)D_t\right)\right)^2 = E_{P^*}\left(J^2(t, s)\right)E_{P^*}\left(D_t^2\right)$$

Using the standard properties of the $It\hat{o}$ integral, we deduce from this that

$$\left(\int_0^t \sigma(u,t)\cdot\sigma(u,s)du\right)^2 = \int_0^t |\sigma(u,t)|^2 du \int_0^t |\sigma(u,s)|^2 du$$

Hence the $\mathbb{R}^d$-valued function $\sigma(.,t)$ and $\sigma(.,s)$ are collinear for any fixed $0 < t \le s \le T^*$. This implies that we have, for any arbitrary $T \in [0, T^*]$

$$\sigma(t,T) = g(T)\sigma(t,T^*), \qquad \forall t \in [0,T]$$

for some function $g:[0,T^*] \to \mathbb{R}$. Upon setting, $h(t) = \sigma(t,T^*),\ t \in \lfloor 0, T^* \rfloor$, we obtain (1.23). For the second statement, it is sufficient to note that, for any $T \le T^*$ and $t \in [0,T]$,

$$b(t,T) = -\int_t^T \sigma(t,u)du = -h(t)\int_t^T g(u)du$$

If, in addition, the coefficient $\sigma^i(t,T)$ is time-homogeneous then

$$Z^i(T-t) \overset{def}{=} \ln\sigma^i(t,T) = \ln g(T) + \ln h^i(t) = \tilde{g}(T) + \tilde{h}^i(t)$$

By differentiating separately with respect to t and T, we find that

$$\left(\tilde{h}^i\right)'(t) = \left(z^i\right)'(T-t), \qquad \tilde{g}'(T) = \left(Z^i\right)'(T-t)$$

and thus

$$\left(Z^i\right)'(T-t) = -\left(\tilde{h}^i\right)'(t) = \tilde{g}'(T) = const. \qquad \text{...(1.26)}$$

It is now easy to check that (1.26) implies (1.24). □

This typical example considers a different case of the volatility structure.

Example 1.13: Let us assume that the volatility of each forward rate is constant, i.e., independent of the maturity date and the level of the forward interest rate. Taking $d = 1$, we thus have $\sigma(t, T) = \sigma$ for a strictly positive constant $\sigma > 0$. By virtue of Equation (1.10), the dynamics of the forward rate process $f(t, T)$ under the martingale measure are given by the expression

$$df(t,T) = \sigma^2(T-t)dt + \sigma dW_t^* \qquad \text{...(1.27)}$$

So that the dynamics of the bond price $B(t, T)$ are

$$dB(t,T) = B(t,T)\left(r_t dt - \sigma(T-t)dW_t^*\right)$$

where the short-term rate of interest r satisfies

$$r_t = f(0,t) + \frac{1}{2}\sigma^2 t^2 + \sigma W_t^*$$

It follows from the last formula that

$$dr_t = \left(f_T(0,t) + \sigma^2 t\right)dt + \sigma dW_t^*$$

Since this agrees with the general form of the continuous-time Ho and Lee model, we conclude that in the HJM framework, the Ho and Lee model corresponds to the constant volatility of forward rates. Dynamics (1.27) make apparent that the only possible movements of the yield curve in the Ho and Lee model are parallel shifts; that is, all rates along the yield curve fluctuate in the same way. For the price $B(t, T)$ of a bond maturity at T, we have that

$$B(t, T) = \frac{B(0, T)}{B(0, t)} \exp\left(-\frac{1}{2}(T-t)Tt\sigma^2 - (T-t)\sigma W_t^* \right) \qquad ...(1.28)$$

and it follows from Equation (1.28) that yields on bonds of differing maturities are perfectly correlated. It can also be expressed in terms of r namely

$$B(t, T) = \frac{B(0, T)}{B(0, t)} \exp\left((T-t) f(0, t) - \frac{1}{2} t (T-t)^2 \sigma^2 - (T-t) r_t \right)$$

Example 1.14: it is a conventional wisdom that forward rates of longer maturity fluctuate less than rates of short maturity. To account for this feature in the HJM framework, we now assume that the volatility of a forward rate is a decreasing function of the time to its effective date. For instance, we may assume that the volatility structure is exponentially dampened, specifically, $\sigma(t, T) = \sigma e^{-b(T-t)}$ for strictly positive real numbers $\sigma, b > 0$.

Then $\sigma^*(t, T)$ is equal to

$$\sigma^*(t, T) = \int_t^T \sigma e^{-b(u-T)} du = -\sigma b^{-1} \left(e^{-b(T-t)} - 1 \right) \qquad ...(1.29)$$

and consequently

$$df(t, T) = -\sigma^2 b^{-1} e^{-b(T-t)} \left(e^{-b(T-t)} - 1 \right) dt + \sigma e^{-b(T-t)} dW_t^* \qquad ...(1.30)$$

It is thus clear that for any maturity T the bond price $B(t, T)$ satisfies

$$dB(t, T) = B(t, T)\left(r_t dt + \sigma b^{-1} \left(e^{-b(T-t)} - 1 \right) dW_t^* \right) \qquad ...(1.31)$$

Substituting Equation (1.29) into Equation (1.11), we obtain

$$r_t = f(0, t) - \int_0^t \sigma^2 b^{-1} e^{-b(t-u)} \left(e^{-b(t-u)} - 1 \right) du + \int_0^t \sigma e^{-b(t-u)} dW_u^*$$

So that, as in the previous example, the negative values of the short-term interest rate are not exclude. Denoting

$$m(t) = f(0, t) + \frac{\sigma^2}{2b^2}\left(e^{-2bt} - 1 \right) - \frac{\sigma^2}{b^2}\left(e^{-bt} - 1 \right),$$

We arrive at the following formula

$$r_t = m(t) + \int_0^t \sigma e^{-b(t-u)} dW_u^*,$$

So that finally

$$dr_t = \left(a(t) - b r_t \right) dt + \sigma dW_t^* \qquad ...(1.32)$$

where $a(t) = bm(t) + m'(t)$. This means that (1.29) leads to a generalized version of Vasicek's model. Note that in the present framework, the perfect fit of the initial term structure is a trivial consequence.

Example 1.15: By combining the models examined in preceding examples, we arrive at the Ho and Lee/Vasicek two-factor version of the Gaussian HJM model, originally put forward by Heath *et al.* (1992a). We assume that the underlying Brownian motion is two-dimensional, and the volatility coefficient σ has two deterministic components σ_1 and σ_2 that correspond to Ho and Lee and Vasicek's models respectively. More explicitly, for any maturity $T \le T^*$ we put

$$\sigma(t, T) = \left(\sigma_1, \sigma_2 e^{-b(T-t)} \right), \qquad \forall t \in [0, T]$$

where σ_1 and σ_2 and γ are strictly positive constants. Similarly, if σ equals

$$\sigma(t, T) = \left(\sigma_1 e^{-b_1(T-t)}, \sigma_2 e^{-b_2(T-t)}\right), \qquad \forall t \in [0, T]$$

for some strictly positive constants σ_1, σ_2, b_1 and b_2, the model is referred to as the two-factor Vasicek model . Closed-formexpressions for the prices of bonds and bond options also easily available in both cases. Since general valuation formulas for the Gaussian HJM set-up are directly applicable to the present models.

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Cite this article as: Hamdin Alafif (2022). Theory of Pricing in Stochastic Financial Models – Continuous Tim. *International Journal of Pure and Applied Mathematics Research*, 2(1), 49-64. doi: 10.51483/IJPAMR.2.1.2022.49-64.