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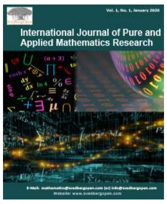

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The Derivation of Various Arbitrary Parameters in the Formulation of Classical Fourth-Order Runge-Kutta Method

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Abstract

Many practical problems in engineering and science are formulated by Ordinary Differential Equations (ODE) that require their own numerical solution. Numerous methods, e.g., the Euler method, the modified Euler method, Heun's method, the Adam-Bashforth method and so on, exist in the context of numerical analysis. Amongst them, the classical Runge-Kutta method (RK4) of the fourth order is mostly used. In this paper, we derive the value of different parameters in the formulation of the fourth order Runge-Kutta method explicitly. The determination techniques are shown stepwise in a straight-forward way. Basically, this paper provides a survey of previous work on deriving the fourth-order Runge-Kutta formula. The major goal of this paper is to provide more details on how to formulate the RK4 method explicitly in order to encourage further research into this method.

Keywords: *Runge-Kutta method, Euler method, Heun's method*

1. Introduction

In numerical analysis, the Runge-Kutta methods were proposed by two German mathematicians, Curl Runge and Wilhelm Kutta, around 1900 (Butcher, 1996). C. Runge published a paper in 1895 which was an extension of the approximation of the Euler method in a more elaborate way. Various order Runge-Kutta methods have been used widely to find the numerical solution of differential equations (Butcher, 2008; Chapra and Canale, 2012). A new version of the improved Runge-Kutta Nystrom method is applied to solve second-order fuzzy differential equations (Parandin, 2013). The Runge-Kutta method of order five is used for the numerical solution of n^{th} order fuzzy differential equations based on the Seikkala derivative with initial value problem (Abbasbandy *et al.*, 2011; Jayakumar *et al.*, 2012; Akbarzadeh and Mohseni, 2011; and Jayakumar *et al.*, 2015). Four and fifth-order Runge-Kutta methods are applied to solve the Lorenz equation (Emre, 2005; Fae'q and Radwan, 2002; and Nikolaos, 2009). Implicit and different multistep Runge-Kutta methods are studied in (Butcher, 1986; Butcher, 1964; and Burrage *et al.*, 1980). The fundamental principles of the theory of differential equations and their numerical solution are discussed by Euler and Coriolis (Euler, 1768; Curtiss and Hirschfelder , 1952). The early works of the Runge-Kutta method are discussed in the papers by Runge (Runge, 1895), Kutta (Kutta, 1901) and Nystrom (Nyström, 1925). The fundamentals of multistep Runge-Kutta methods are published

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by Adams and Bashforth, Dahlquist and Moulton (Dahlquist, 1956 and 1963). Recently, some work on the Runge-Kutta method is seen, e.g., Mechee and Yasen (2017), Geeta and Varun (2020) applied generalized RK integrators for solving ordinary differential equations. Computational techniques based on the Runge-Kutta method of various orders and types for solving differential equations are discussed by Vijeyata and Pankaj (2019). Explicit fourth-derivative two-step linear multistep methods have been studied by Wusu and Akanbi (2017) to solve ordinary differential equations.

From this brief survey, it is clear that Runge-Kutta methods, as well as their various versions, are used to solve differential equations in a variety of fields. The formulas for the Runge-Kutta method can be found in most textbooks (Goel and Mittal, 1998; Jain and Iyenger, 2014). Some of the books and publications provide the Runge-Kutta method in a broader context and do not go into depth about the approach in its ultimate form (Chapra and Canale, 2012). Butcher presents the parameter values of the Runge-Kutta method in a tableau, but in a compact manner (Butcher, 2008). The goal of this study is to describe in detail how to derive various arbitrary parameters using the most widely used fourth order Runge-Kutta method. The majority of the information in this publication is sourced mostly from (Butcher, 2008). The paper is organized as follows: Introduction is given in Section 1. A detailed description of calculating the different parameters in the formation of fourth order Runge-Kutta method is given in Section 2. Sections 3 through 4 contain the discussion and conclusion.

2. Derivation of the Parameters of Fourth Order Runge-Kutta (RK4) Method

The basic idea of fourth order Runge-Kutta method is to find the numerical solution of the differential equation

$$y' = f(x, y), \ \ y(x_0) = y_0 \qquad ...(1)$$

There exists different form of Runge-Kutta method but all can be put in the generalized form as:

$$y_{m+1} = y_m + hf(x_m, y_m) \qquad ...(2)$$

where $f(x_m, y_m)$ is an increment function which can be written in general form as:

$$f = a_1k_1 + a_2k_2 + a_3k_3 + \cdots + a_nk_n \qquad ...(3)$$

The a_i's and k_i's in Equation (3) are arbitrary constants where k_i's are given by:

$$\begin{aligned} k_1 &= f(x_m, y_m) \\ k_2 &= f(x_m + u_1h, y_m + v_{11}k_1h) \\ k_3 &= f(x_m + u_2h, y_m + v_{21}k_1h + v_{22}k_2h) \\ &\vdots \\ k_n &= f(x_m + u_{n-1}h, y_m + v_{n-1,1}k_1h + v_{n-2,2}k_2h + \cdots + v_{n-1,n-1}k_{n-1}h) \end{aligned} \qquad ...(4)$$

It is clear from Equation (4) that each is a functional evaluation and k_i's is in recurrence relationship. A more used general form of the fourth order Runge-Kutta method of Equations (3) and (4) is given by:

$$y(x+h) \approx y(x) + ak_1 + bk_2 + ck_3 + dk_4 \qquad ...(5)$$

where k_i's are

$$\begin{aligned} k_1 &= hf(x, y) \\ k_2 &= hf(x + mh, y + mk_1) \\ k_3 &= hf(x + nh, y + nk_2) \\ k_4 &= hf(x + ph, y + pk_3) \end{aligned} \qquad ...(6)$$

We derive the arbitrary constants a, b, c, d, m, n, p such that Equation (5) agrees with the Taylor series solution up to the term h^4.

From Equation (1) we get

$$y' = \frac{dy}{dx} = f(x,y) = f \qquad ...(7)$$

Differentiating Equaiton (7), we derive

$$y'' = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{df(x,y)}{dx} = \frac{\partial f}{\partial x}\cdot\frac{dx}{dx} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dx} \qquad ...(8)$$

which implies that

$$\begin{aligned} y'' &= f_x + f.f_y \\ &= E_1 \end{aligned} \qquad ...(9)$$

where $E_1 = f_x + f.f_y$.

Differentiating Equation (9) another times

$$\begin{aligned} y''' &= \frac{d}{dx}(y'') = \frac{d}{dx}\left(f_x(x,y) + f(x,y).f_y(x,y)\right) \\ &= \frac{df_x(x,y)}{dx} + f(x,y).\frac{df_y(x,y)}{dx} + f_y(x,y).\frac{df(x,y)}{dx} \\ &= \frac{\partial f_x}{\partial x}\cdot\frac{dx}{dx} + \frac{\partial f_x}{\partial y}\cdot\frac{dy}{dx} + f.\left[\frac{\partial f_y}{\partial x}.\frac{dx}{dx} + \frac{\partial f_y}{\partial y}\cdot\frac{dy}{dx}\right] + f_y\left(f_x + f.f_y\right) \\ &= f_{xx}\cdot 1 + f_{xy}\cdot f + f\left[f_{yx}\cdot 1 + f_{yy}\cdot f\right] + f_y.f_x + f.f_y^2 \end{aligned}$$

which implies that

$$y''' = (f_{xx} + 2f.f_{xy} + f^2.f_{yy}) + f_y(f_x + f.f_y) \qquad ...(10)$$

Let us consider $E_2 = f_{xx} + 2f.f_{xy} + f^2.f_{yy}$, thus (10) becomes

$$y''' = E_2 + f_y E_1 \qquad ...(11)$$

Again differentiating Equation (10), we derive

$$\begin{aligned} y^{(iv)} &= \frac{d}{dx}(y''') = \frac{d}{dx}[(f_{xx} + 2f.f_{xy} + f^2.f_{yy}) + f_y(f_x + f.f_y)] \\ &= \frac{d}{dx}(f_{xx} + 2f.f_{xy} + f^2.f_{yy} + f_x.f_y + f.f_y^2) \\ &= \frac{d}{dx}(f_{xx}) + 2\frac{d}{dx}(f.f_{xy}) + \frac{d}{dx}(f^2.f_{yy}) + \frac{d}{dx}(f_y.f_x) + \frac{d}{dx}(f.f_y^2) \\ &= \frac{\partial f_{xx}}{\partial x}\frac{dx}{dx} + \frac{\partial}{\partial y}(f_{xx}).\frac{dy}{dx} + 2\left[f_{xy}.\frac{df}{dx} + f.\frac{d}{dx}(f_{xy})\right] + \left[f_{yy}.2f.f' + f^2.\frac{d}{dx}(f_{yy})\right] \\ &\qquad + \left[f_y\frac{d}{dx}(f_x) + f_x\frac{d}{dx}(f_y)\right] + \left[f_y^2.\frac{df}{dx} + 2f.f_y\frac{d}{dx}(f_y)\right] \end{aligned}$$

$$\begin{aligned} &= [f_{xxx}.1 + f_{xxy}.f] + 2\left[f_{xy}(f_x + f.f_y) + f\left(\frac{\partial f_{xy}}{\partial x}.\frac{dx}{dx} + \frac{\partial f_{xy}}{\partial y}.\frac{dy}{dx}\right)\right] + \left[2f(f_x + f.f_y) + f^2\left(\frac{\partial f_{yy}}{\partial x}.\frac{dx}{dx} + \frac{\partial f_{yy}}{\partial y}.\frac{dy}{dx}\right)\right] \\ &\qquad + \left[f_y\left(\frac{\partial f_x}{\partial x}.\frac{dx}{dx} + \frac{\partial f_x}{\partial y}.\frac{dy}{dx}\right) + f_x\left(\frac{\partial f_y}{\partial x}.\frac{dx}{dx} + \frac{\partial f_y}{\partial y}.\frac{dy}{dx}\right)\right] + \left[f_y^2\left(f_x + f.f_y\right) + 2f.f_y\left(\frac{\partial f_y}{\partial x}.\frac{dx}{dx} + \frac{\partial f_y}{\partial y}.\frac{dy}{dx}\right)\right] \end{aligned}$$

$$=[f_{xxx}+f.f_{xxy}]+\left[2f_x f_{xy}+2f.f_y.f_{xy}+2f(f_{xxy}+f_{xyy}.f)\right]$$
$$+\left[(2ff_x+2f^2.f_y)f_{yy}+f^2(f_{xyy}+f.f_{yyy})\right]+\left[f_y(f_{xx}+f_{xy}.f)+f_x(f_{xy}+f_{yy}.f)\right]$$
$$+\left[f_x.f_y^2+f.f_y^3+2f.f_y(f_{xy}+f.f_{yy})\right]$$
$$=f_{xxx}+3f.f_{xxy}+3f^2f_{xyy}+f^3f_{yyy}+f_yf_{xx}+2f.f_yf_{xy}+f^2f_yf_{yy}+3f_xf_{xy}+3f.f_y.f_{xy}+3f.f_x.f_{yy}$$
$$+3f^2f_yf_{yy}+f_xf_y^2+f.f_y^3$$
$$=\left(f_{xxx}+3f.f_{xxy}+3f^2f_{xyy}+f^3f_{yyy}\right)+f_y\left(f_{xx}+2f.f_{xy}+f^2f_{yy}\right)+3\left(f_x+f.f_y\right)\left(f_{xy}+f.f_{yy}\right)+f_y^2\left(f_x+f.f_y\right)$$

Therefore,

$$y^{(iv)}=\left(f_{xxx}+3f.f_{xxy}+3f^2f_{xyy}+f^3f_{yyy}\right)+f_y\left(f_{xx}+2f.f_{xy}+f^2f_{yy}\right)+3\left(f_x+f.f_y\right)\left(f_{xy}+f.f_{yy}\right)$$
$$+f_y^2\left(f_x+f.f_y\right) \qquad ...(12)$$

Again let $E_3=\left(f_{xxx}+3f.f_{xxy}+3f^2f_{xyy}+f^3f_{yyy}\right)$, thus Equation (12) becomes

$$y^{(iv)}=E_3+f_yE_2+3E_1\left(f_{xy}+f.f_{yy}\right)+f_y^2E_1 \qquad ...(13)$$

Now, the Taylor's series is,

$$y\left(x+h\right)=y(x)+hy'(x)+\frac{h^2}{2!}y''(x)+\frac{h^3}{3!}y'''(x)+\frac{h^4}{4!}y^{(iv)}(x)+O\left(h^5\right) \qquad ...(14)$$

Putting the values of $y'(x), y''(x), y'''(x)$ and $y^{(iv)}(x)$ in Equation (14), we get

$$y\left(x+h\right)=y(x)+hf+\frac{h^2}{2}E_1+\frac{h^3}{6}\left(E_2+f_yE_1\right)+\frac{h^4}{24}\left[E_3+f_yE_2+3E_1\left(f_{xy}+ff_{yy}\right)+f_y{}^2E_1\right]+\cdots$$
$$=y(x)+hf+\frac{h^2}{2}E_1+\frac{h^3}{6}E_2+\frac{h^4}{24}E_3+\frac{h^3}{6}f_yE_1+\frac{h^4}{24}f_yE_2+\frac{1}{8}h^4\left(f_{xy}+ff_{yy}\right)E_1+\frac{h^4}{24}f_y{}^2E_1+\cdots \qquad ...(15)$$

Here,

$k_1=h.f(x,y)=h.f$

$k_2=hf(x+mh,y+mk_1)$

Now, expanding the double variable function $f(x+mh,y+mk_1)$ by Taylor's series we get,

$$f(x+mh,y+mk_1)=f(x,y)+\left(mh\frac{\partial}{\partial x}+mk_1\frac{\partial}{\partial y}\right)f(x,y)+\frac{1}{2!}\left(mh\frac{\partial}{\partial x}+mk_1\frac{\partial}{\partial y}\right)^2f(x,y)+\frac{1}{3!}\left(mh\frac{\partial}{\partial x}+mk_1\frac{\partial}{\partial y}\right)^3f(x,y)+\cdots$$
$$=f+\left(mhf_x+mk_1f_y\right)+\frac{1}{2}\left(m^2h^2f_{xx}+2m^2hk_1f_{xy}+m^2k_1^2f_{yy}\right)$$
$$+\frac{1}{6}\left(m^3h^3f_{xxx}+3m^2h^2mk_1f_{xxy}+3mhm^2k_1^2f_{xyy}+m^3k_1^3f_{yyy}\right)+\cdots$$

Substituting $k_1=h.f$ in the above equation

$$=f+mh\left(f_x+f.f_y\right)+\frac{1}{2}m^2h^2\left(f_{xx}+2f.f_{xy}+f^2f_{yy}\right)$$
$$+\frac{1}{6}m^3h^3\left(f_{xxx}+3f.f_{xxy}+3f^2f_{xyy}+f^3f_{yyy}\right)+\cdots$$
$$=f+mhE_1+\frac{1}{2}m^2h^2E_2+\frac{1}{6}m^3h^3E_3+\cdots$$

Therefore,

$$f(x+mh, y+mk_1) = f + mhE_1 + \frac{1}{2}m^2h^2E_2 + \frac{1}{6}m^3h^3E_3 + \cdots \qquad ...(16)$$

Hence,

$$k_2 = h\left[f + mhE_1 + \frac{1}{2}m^2h^2E_2 + \frac{1}{6}m^3h^3E_3 + \cdots \right] \qquad ...(17)$$

Again, given that

$k_3 = hf(x+nh, y+nk_2),$

Further expanding $f(x+nh, y+nk_2)$ in Taylor's series, we get

$$\begin{aligned} f(x+nh, y+nk_2) &= f(x,y) + \left(nh\frac{\partial}{\partial x} + nk_2\frac{\partial}{\partial y} \right) f + \frac{1}{2!}\left(nh\frac{\partial}{\partial x} + nk_2\frac{\partial}{\partial y} \right)^2 f + \frac{1}{3!}\left(nh\frac{\partial}{\partial x} + nk_2\frac{\partial}{\partial y} \right)^3 f + \cdots \\ &= f + nhf_x + nk_2f_y + \frac{1}{2}\left(n^2h^2f_{xx} + 2n^2hk_2f_{xy} + n^2k_2^2f_{yy} \right) \\ &\quad + \frac{1}{6}\left(n^3h^3f_{xxx} + 3n^2h^2nk_2f_{xxy} + 3nhn^2k_2^2f_{xyy} + n^3k_2^3f_{yyy} \right) + \cdots \end{aligned}$$

Now, substituting this value of k_2 in the above equation we get

$$\begin{aligned} f(x+nh, y+nk_2) &= f + nhf_x + \left(nhff_y + mnh^2f_yE_1 + \frac{1}{2}m^2nh^3f_yE_2 + \frac{1}{6}m^3nf_yh^4E_3 + \cdots \right) \\ &\quad + \frac{1}{2}\left[n^2h^2f_{xx} + 2n^2hf_{xy}\left(hf + mh^2E_1 + \frac{1}{2}m^2h^3E_2 + \frac{1}{6}m^3h^4E_3 + \cdots \right) \right. \\ &\quad \left. + n^2f_{yy}\left(hf + mh^2E_1 + \frac{1}{2}m^2h^3E_2 + \frac{1}{6}m^3h^4E_3 + \cdots \right)^2 \right] \\ &\quad + \frac{1}{6}\left[n^3h^3f_{xxx} + 3n^3h^2f_{xxy}\left(hf + mh^2E_1 + \frac{1}{2}m^2h^3E_2 + \frac{1}{6}m^3h^4E_3 + \cdots \right) \right. \\ &\quad + 3n^3hf_{xyy}\left(hf + mh^2E_1 + \frac{1}{2}m^2h^3E_2 + \frac{1}{6}m^3h^4E_3 + \cdots \right)^2 \\ &\quad \left. + n^3f_{yyy}\left(hf + mh^2E_1 + \frac{1}{2}m^2h^3E_2 + \frac{1}{6}m^3h^4E_3 + \cdots \right)^3 \right] \end{aligned}$$

Collecting the orders of $O(h^3)$

$$\begin{aligned} f(x+nh, y+nk_2) &= f + nhF_1 + \frac{1}{2}h^2\left(n^2F_2 + 2mnf_yF_1 \right) + \\ &\quad \frac{1}{6}h^3\left(n^3F_3 + 3m^2nf_yF_2 + 6mn^2\left(f_{xy} + ff_{yy} \right)F_1 \right) + \cdots \end{aligned} \qquad ...(18)$$

Therefore,

$$k_3 = h\left[f + nhF_1 + \frac{1}{2}h^2\left(n^2F_2 + 2mnf_yF_1 \right) + \frac{1}{6}h^3\left(n^3F_3 + 3m^2nf_yF_2 + 6mn^2\left(f_{xy} + ff_{yy} \right)F_1 \right) + \cdots \right] \qquad ...(19)$$

Again, given that

$k_4 = hf\left(x + ph, y + pk_3\right)$

Expanding $f\left(x + ph, y + pk_3\right)$ Taylor's series we get,

$$f(x+ph, y+pk_3) = f(x,y) + \left(ph\frac{\partial}{\partial x} + pk_3\frac{\partial}{\partial y}\right)f + \frac{1}{2!}\left(ph\frac{\partial}{\partial x} + pk_3\frac{\partial}{\partial y}\right)^2 f + \frac{1}{3!}\left(ph\frac{\partial}{\partial x} + pk_3\frac{\partial}{\partial y}\right)^3 f + \cdots$$
$$= f + phf_x + pk_3 f_y + \frac{1}{2}p^2\left(h^2 f_{xx} + 2hk_3 f_{xy} + k_3^2 f_{yy}\right) +$$
$$\frac{1}{6}p^3\left(h^3 f_{xxx} + 3h^2 k_3 f_{xxy} + 3hk_3^2 f_{xyy} + k_3^3 f_{yyy}\right) + \cdots$$

Substituting the value of k_3 in the above equation we get

$$f(x+ph,y+pk_3) = f + phf_x + pf_y\left[hf + nhE_1 + \frac{1}{2}h^3\left(n^2E_2 + 2mnf_yE_1\right) + \frac{1}{6}h^4\left(n^3E_3 + 3m^2nf_yE_2 + 6mn^2\left(f_{xy} + ff_{yy}\right)E_1\right) + \cdots\right]$$
$$+\frac{1}{2}\left[p^2h^2f_{xx} + 2p^2h\left(hf + nhE_1 + \frac{1}{2}h^3\left(n^2E_2 + 2mnf_yE_1\right) + \frac{1}{6}h^4\left(n^3E_3 + 3m^2nf_yE_2 + 6mn^2\left(f_{xy} + ff_{yy}\right)E_1\right) + \cdots\right)\right]$$
$$+\frac{1}{2}\left[p^2f_{yy}\left(hf + nhE_1 + \frac{1}{2}h^3\left(n^2E_2 + 2mnf_yE_1\right) + \frac{1}{6}h^4\left(n^3E_3 + 3m^2nf_yE_2 + 6mn^2\left(f_{xy} + ff_{yy}\right)E_1\right) + \cdots\right)^2\right]$$
$$+\frac{1}{6}\left(p^3h^3f_{xxx} + 3p^2h^2pf_{xxy}\left(hf + nhE_1 + \frac{1}{2}h^3\left(n^2E_2 + 2mnf_yE_1\right) + \frac{1}{6}h^4\left(n^3E_3 + 3m^2nf_yE_2 + 6mn^2\left(f_{xy} + ff_{yy}\right)E_1\right) + \cdots\right) + \cdots\right)$$

Now on simplification

$$f(x+ph, y+pk_3) = f + phE_1 + \frac{1}{2}h^2\left(p^2E_2 + 2pnf_yE_1\right) +$$
$$\frac{1}{6}h^3\left(p^3E_3 + 3n^2pf_yE_2 + 6np^2\left(f_{xy} + ff_{yy}\right)E_1 + 6mnpf_y^2E_1\right) + \cdots \quad ...(20)$$

Now, substituting this value in the $k_4 = hf\left(x + ph, y + pk_3\right)$ equation we get,

$$k_4 = h\left[f + phE_1 + \frac{1}{2}h^2\left(p^2E_2 + 2npf_yE_1\right) + \frac{1}{6}h^3\left(p^3E_3 + 3n^2pf_yE_2 + 6np^2\left(f_{xy} + ff_{yy}\right)E_1 + 6mnpf_y^2E_1\right) + \cdots\right] \quad ...(21)$$

Substituting the values of k_1, k_2, k_3, k_4 in Equation (5) we get,

$$y(x+h) \approx y(x) + (a+b+c+d)hf + (bm+cn+dp)hE_1 + \frac{1}{2}\left(bm^2 + cn^2 + dp^2\right)h^3E_2$$
$$+\frac{1}{6}\left(bm^3 + cn^3 + dp^3\right)h^4E_3 + \left(cmn + dnp\right)h^3f_yE_1 + \frac{1}{2}\left(cm^2n + dn^2p\right)h^4f_yE_2 \quad ...(22)$$
$$+\left(cmn^2 + dnp^2\right)h^4\left(f_{xy} + ff_{yy}\right)E_1 + dmnph^4f_y^2E_1 + O(h^5)$$

Comparing the eEuations (15) and (22) we get

$$a + b + c + d = 1$$
$$bm + cn + dp = \frac{1}{2}$$

$$
\begin{aligned}
a+b+c+d &= 1 \\
bm+cn+dp &= \frac{1}{2} \\
bm^2+cn^2+dp^2 &= \frac{1}{3} \\
bm^3+cn3+dp^3 &= \frac{1}{4} \\
cmn+dnp &= \frac{1}{6} \\
cmn^2+dnp^2 &= \frac{1}{8} \\
cm^2n+dn^2p &= \frac{1}{12} \\
dmnp &= \frac{1}{24}
\end{aligned}
\qquad ...(23)
$$

Equation (23) is an overdetermined system of eight equations and seven variables. Using the following Maple command

$$RK\_Solution := solve(\{a+b+c+d=1,\ b.m+c.n+d.p=\frac{1}{2}, b.m^2+c.n^2+d.p^2=\frac{1}{3}, b.m^3+c.n^3+d.p^3=\frac{1}{4}, c.m.n+d.n.p=\frac{1}{6}, c.m.n^2+d.n.p^2=\frac{1}{8}, c.m^2.n+d.n^2.p=\frac{1}{12}, d.m.n.p=\frac{1}{24}\}, \{a,b,c,d,m,n,p\});$$

and the classical solution is

$$RK\_Solution := \{a=\frac{1}{6}, b=\frac{1}{3}, c=\frac{1}{3}, d=\frac{1}{6}, m=\frac{1}{2}, n=\frac{1}{2}, p=1\} \qquad ...(24)$$

Putting these values in the Equations (5) and (6) we get,

$$y(x+h)=y(x)+\frac{1}{6}(k_1+2k_2+2k_3+k_4), \qquad ...(25)$$

where,

$$
\begin{aligned}
k_1 &= hf(x,y) \\
k_2 &= hf\left(x+\frac{h}{2}, y+\frac{1}{2}k_1\right) \\
k_3 &= hf\left(x+\frac{h}{2}, y+\frac{1}{2}k_2\right) \\
k_4 &= hf(x+h, y+k_3).
\end{aligned}
\qquad ...(26)
$$

3. Discussion

There are numerous Runge-Kutta procedures have been posed and developed. In the literature, there have been several theoretical and numerical investigations dealing with the solution of differential equations of various orders. Amongst them, the Runge-Kutta methods are a set of implicit and explicit methods for approximating the solutions of ODEs in numerical analysis. This reason motivates us to study and derive the RK methods more directly. Notice that in Equation (2) the increment functions $f(x_m, y_m)$ is utilized and the *k*'s in Equation (4) are in recurrence relationships.

The Equations (5) and (6) exhibits of using a weighted approximations of where the weights are a's and m, n, p are arbitrary constants. The derivation procedures of these constants are shown unambiguously stepwise. The mathematical computational technology Maple is used to find the solution of the overdetermined system of equations.

4. Conclusion

The main purpose of this work is to provide more details about formulating the fourth order Runge-Kutta method in order to motivate the study of this method more deeply. Our goal in writing this review is to have a better understanding of the underlying principality of the RK method, as well as enhance its analytical capabilities. The work is particularly important for understanding the overall formulation of the fourth order Runge-Kutta method.

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