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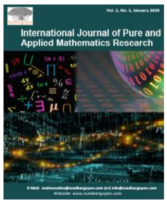

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Soliton Types Wave Solutions to Fractional Order Nonlinear Evolution Equations Arise in Mathematical Physics

Md. Tarikul Islam[1*] and Mst. Armina Akter[2]

[1]Department of Mathematics, Hajee Mohammad Danesh Science and Technology University, Dinajpur, Bangladesh. E-mail: tarikul_hstu@yahoo.com

[2]Department of Mathematics, Hajee Mohammad Danesh Science and Technology University, Dinajpur, Bangladesh. E-mail: aaktermath@gmail.com

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Abstract

Fractional order Nonlinear Evolution Equations (FNLEEs) concerning to conformable fractional derivative bears great importance in various fields of real world as the model to describe underling mechanisms of nature. In this paper, we make known a new technique, called the modified fractional generalized $\left(G'/G^2\right)$ -expansion method, to study the nonlinear space-time fractional mKdV equation and the nonlinear space-time fractional SRLW equation. A compound wave variable transformation reduces the considered equations to ordinary differential equations. Then the proposed method is employed to construct their solutions. The obtained solutions in terms of trigonometric function, hyperbolic function and rational function are claimed to be fresh and further general in closed form. These solutions might play important roles to depict the complex physical phenomena arise in nature. The modified fractional generalized $\left(G'/G^2\right)$-expansion method shows high performance and might be used as a strong tool to unravel any other FNLEEs.

Keywords: *The modified fractional generalized $\left(G'/G^2\right)$-expansion method; compound wave variable transformation, Conformable fractional derivative, Closed form solution, Fractional order nonlinear evolution equation*

1. Introduction

Fractional calculus originating from some speculations of Leibniz and L'Hospital in 1695 has gradually gained profound attention of many researchers for its extensive appearance in various fields of real world. The Fractional order Nonlinear Evolution Equations (FNLEEs) and their solutions in closed form play fundamental role in describing, modeling and predicting the underlying mechanisms related to the biology, bio-genetics, physics, solid state physics, condensed matter physics, plasma physics, optical fibers, meteorology, oceanic phenomena, chemistry, chemical kinematics, electromagnetic, electrical circuits, quantum mechanics, polymeric materials, neutron point kinetic model, control and vibration, image and signal processing, system identifications, the finance, acoustics and fluid dynamics (Oldham and Spanier, 1974; Samko *et al.*, 1993; Podlubny, 1999). The closed form wave solutions of these equations (Mainardi, 2010; Baleanu *et al.*, 2012; Yang, 2012) are greatly helpful to understand the mechanisms of the phenomena as well as their

* *Corresponding author: Md. Tarikul Islam, Department of Mathematics, Hajee Mohammad Danesh Science and Technology University, Dinajpur, Bangladesh. E-mail: tarikul_hstu@yahoo.com*

further application in practical life. Some attractive powerful approaches take into account in the recent research area related to fractional derivative associated problems (He *et al.*, 2012; He and Ji, 2019 and 2020). Therefore, it has become the core aim in the research area of fractional related problems that how to develop a stable approach for investigating the solutions to FNLEEs in analytical or numerical form. Many researchers have offered different approaches to construct analytic and numerical solutions to NLEEs of fractional order as well as integer order and put them forward for searching traveling wave solutions, such as the He-Laplace method (Li and Nadeem, 2019), the exponential decay law (Atangana and Aguilar, 2017), the reproducing kernel method (Akgul *et al.*, 2017), the Jacobi elliptic function method (Aslan and Inc, 2017), the (G'/G^2)-expansion method and its various modifications (Baleanu *et al.*, 2015; Inan *et al.*, 2015; Islam *et al.*, 2018a; 2018b; and 2018c), the Exp-function method (Guner *et al.*, 2015), the sub-equation method (Alzaidy, 2013), the first integral method (Martinez *et al.*, 2018), the functional variable method (Inc *et al.*, 2017), the modified trial equation method (Bulut *et al.*, 2013), the simplest equation method (Taghizadeh *et al.*, 2013), the Lie group analysis method (Chen and Jiang, 2015), the fractional characteristic method (Wu, 2011), the auxiliary equation method (Seadawy, 2017; and Akbulut *et al.*, 2016), the finite element method (Deng, 2008), the differential transform method (Momani *et al.*, 2007), the Adomian decomposition method (Hu *et al.*, 2008; and El-Sayed *et al.*, 2010), the variational iteration method (Inc, 2008), the finite difference method (Gao *et al.*, 2012), the homotopy perturbation method (Gepreel, 2011) and the He's variational principle (Inc, 2013), etc. But no method is uniquely substantial to examine the closed form solutions to all kind of FNLEEs. That is why; it is very much indispensable to establish new techniques.

In this paper, we propose a new technique, called the modified fractional generalized (G'/G^2)-expansion method, to construct closed form analytic wave solutions to some FNLEEs in the sense of conformable fractional derivative (Khalil *et al.*, 2014). This effectual and reliable productive method shows its high performance through providing abundant fresh and general solutions to the suggested equations. The obtained solutions might bring up their importance through the contribution to analyze the inner mechanisms of physical complex phenomena of real world and make an acceptable record in the literature.

2. Preliminaries and Methodology

2.1. Conformable Fractional Derivative

A new and simple definition of derivative for fractional order introduced by Khalil *et al.* (2014) is called conformable fractional derivative. This definition is analogous to the ordinary derivative

$$\frac{d\psi}{dx}=\lim_{\varepsilon\to 0}\frac{\psi(x+\varepsilon)-\psi(x)}{\varepsilon},$$

where $\psi(x):[0,\infty]\to\mathbf{R}$ and $x>0$. According to this classical definition, $\frac{d(x^n)}{dx}=nx^{n-1}$. According to this perception, Khalil has introduced α order fractional derivative of ψ as

$$T_\alpha\psi(x)=\lim_{\varepsilon\to 0}\frac{\psi(x+\varepsilon x^{1-\alpha})-\psi(x)}{\varepsilon},\ 0<\alpha\le 1$$

If the function ψ is α-differentiable in $(0,r)$ for $r>0$ and $\lim_{x\to 0^+}T_\alpha\psi(x)$ exists, then the conformable derivative at $x=0$ is defined as $T_\alpha\psi(0)=\lim_{x\to 0^+}T_\alpha\psi(x)$. The conformable integral of ψ is

$$I_\alpha^r\psi(x)=\int_r^x\frac{\psi(t)}{t^{1-\alpha}}dt,\ r\ge 0,0<\alpha\le 1$$

This integral represents usual Riemann improper integral.

The conformable fractional derivative satisfies the following useful properties (Khalil *et al.*, 2014):

If the functions $u(x)$ and $v(x)$ are α-differentiable at any point $x>0$, for $\alpha\in(0,1]$, then

(a) $T_\alpha(au+bv) = aT_\alpha(u)+bT_\alpha(v) \;\; \forall a,b \in \mathbf{R}$.

(b) $T_\alpha(x^n) = n\,x^{n-\alpha} \;\; \forall n \in \mathbf{R}$.

(c) $T_\alpha(c) = 0$, where is any constant.

(d) $T_\alpha(uv) = uT_\alpha(v)+vT_\alpha(u)$.

(e) $T_\alpha(u/v) = \dfrac{vT_\alpha(u)-uT_\alpha(v)}{v^2}$.

(f) If u is differentiable, then $T_\alpha(u)(x) = x^{1-\alpha}\dfrac{du}{dx}(x)$.

Many researchers used this new derivative of fractional order in physical applications due to its convenience, simplicity and usefulness (Atangana *et al.*, 2015; Cenesiz and Kurt, 2015; and Eslami and Rezazadeh, 2016).

2.2. Methodology

In this section, we discuss the main steps of the above-mentioned method to investigate exact analytic solutions of FNLEEs. Consider the FNLEE in the independent variables $t, x_1, x_2, \ldots, x_n$ as

$$F\left(u_1, \ldots, u_k, D_t^\alpha u_k, D_{x_1}^\beta u_1, \ldots, D_{x_1}^\beta u_k, \ldots, D_{x_n}^\beta u_1, \ldots, D_{x_n}^\beta u_k, \ldots\right) = 0 \qquad \ldots(2.2.1)$$

where $u_i = u_i(t, x_1, x_2, \ldots, x_n),\; i = 1, \ldots, k$ are unknown functions, F is a polynomial in u_i and it's various partial derivatives of fractional order.

Making use of the composite wave variable transformation

$$u_i = u_i(t, x_1, x_2, \ldots, x_n) = U_i(\xi),\; \xi = \xi(t, x_1, x_2, \ldots, x_n) \qquad \ldots(2.2.2)$$

Equation (2.2.1) is turned into the following ordinary differential equation with respect to the variable ξ:

$$Q(U, U', U'', U''', \ldots) = 0 \qquad \ldots(2.2.3)$$

where Q is a polynomial of U and its derivatives and the superscripts indicates ordinary derivatives with respect to ξ.

We may, if possible, take the anti-derivative of Equation (2.2.3) term by term one or more times and integral constant can be set to zero as soliton solutions are sought. Then the offered method is employed to construct closed form analytic solutions of Equation. (2.2.3).

The main steps of the modified fractional generalized $\left(G'/G^2\right)$-expansion method is discussed for finding exact analytic solutions to FNLEEs.

Step 1: Consider the solution of Equation (2.2.3) as follows:

$$U(\xi) = \sum_{i=0}^{n} a_i \Phi^i + \sum_{i=1}^{n} b_i \Phi^{-i} \qquad \ldots(2.2.4)$$

where $\Phi = \varepsilon + \left(G'/G^2\right)$.

where a_i and $b_i (i = 1, 2, \ldots, n)$ are arbitrary constants to be determined later with at least one of a_n and b_n as non-zero and $G = G(\xi)$ satisûes the succeeding second order ordinary differential equation:

$$G''G^2 - 2GG'^2 = \rho\left(G'\right)^2 + \mu G'G^2 + \sigma G^4 \qquad \ldots(2.2.5)$$

where ρ, σ and μ are real constants. Equation (2.2.5) has turned into

$$\left(G'/G^2\right)' = \rho\left(G'/G^2\right)^2 + \mu\left(G'/G^2\right) + \sigma \qquad \ldots(2.2.6)$$

Then we have the general solutions of Equation (2.2.5) (or equivalent to Equation (2.2.6) as follows:

$$\left(G'/G^2\right)=\begin{cases}\dfrac{\sqrt{\rho\sigma}\left(A\cos\left(\sqrt{\rho\sigma}\xi\right)+B\sin\left(\sqrt{\rho\sigma}\xi\right)\right)}{\sigma\left(B\cos\left(\sqrt{\rho\sigma}\xi\right)-A\sin\left(\sqrt{\rho\sigma}\xi\right)\right)}, & if\ \rho\sigma>0,\ u=0\\ \dfrac{-\sqrt{|\rho\sigma|}\left(A\sin h\left(2\sqrt{|\rho\sigma|}\xi\right)+A\cos h\left(2\sqrt{|\rho\sigma|}\xi\right)+B\right)}{\sigma\left(A\sin h\left(2\sqrt{|\rho\sigma|}\xi\right)+A\cos h\left(2\sqrt{|\rho\sigma|}\xi\right)-B\right)}, & if\ \rho\sigma>0,\ \mu=0\\ \dfrac{-A}{\rho\left(A\xi+B\right)}, & if\ \sigma=0,\ \rho\neq 0,\ \mu=0\\ \dfrac{-\mu}{2\rho}-\dfrac{\sqrt{\Delta}\left(A\cos h\left(\sqrt{\Delta}/2\xi\right)+B\sin h\left(\sqrt{\Delta}/2\xi\right)\right)}{2\rho\left(B\cos h\left(\sqrt{\Delta}/2\xi\right)+A\sin h\left(\sqrt{\Delta}/2\xi\right)\right)} & if\ \Delta>0,\ \mu\neq 0\\ \dfrac{-\mu}{2\rho}-\dfrac{\sqrt{-\Delta}\left(A\cos\left(\sqrt{-\Delta}/2\xi\right)+B\sin\left(\sqrt{-\Delta}/2\xi\right)\right)}{2\rho\left(B\cos\left(\sqrt{-\Delta}/2\xi\right)+A\sin\left(\sqrt{-\Delta}/2\xi\right)\right)} & if\ \Delta>0,\ \mu\neq 0\end{cases} \quad ...(2.2.7)$$

where A and B are arbitrary constants and $\Delta = \mu^2 - 4\rho\sigma$.

Step 2: Consider the homogeneous balance between the highest order derivatives and the nonlinear terms appearing in Equation (2.2.3) to determine the positive integer n. If the degree of $U(\xi)$ is deg $[U(\xi)] = n$, then, the degree of other expressions will be

$$\deg\left[\frac{d^nU(\xi)}{d\xi^m}\right]=n+m,\ \deg\left[U^m\left(\frac{d^lU(\xi)}{d\xi^l}\right)^p\right]=mn+p(n+l)$$

Step 3: Substituting Equation (2.2.4) as well as Eq. (2.2.5) into Equation (2.2.3), we get a polynomial of $\left(G'/G^2\right)$, in which we equate all the coefficients to zero. This procedure yields a system of algebraic equations which can be solved for getting a_i, b_i, ρ, σ and μ as well as the values of the other necessary parameters.

Step 4: We substitute the values of a_i, b_i, ρ, σ and μ together with the solutions given in Equation (2.2.7) into Equation (2.2.4). This completes the determination of the solutions to the nonlinear evolution Equation (3.2.1).

3. Formulation of the Solutions

In this section, the suggested method is applied to unravel the space-time fractional mKdV equation and the space-time fractional SRLW equation for their analytic solutions in closed form.

3.1. The Space-Time Fractional mKdV Equation

This well-known equation has the form

$$D_t^{\alpha}u+\eta u^2D_x^{\alpha}u+\tau D_t^{3\alpha}u=0,\ 0<\alpha\leq 1 \quad ...(3.1.1)$$

Consider the wave variable transformation as

$$u(x,t)=U(\xi),\ \xi=kx+ct \quad ...(3.1.2)$$

Equation (3.1.1) with the aid of Equation (3.1.2) is turned into the following ordinary differential equation due to the variable ξ:

$$cU'+k\mu U^2U'+k^3\tau U'''=0 \quad ...(3.1.3)$$

Taking anti-derivative of Equation (3.1.3) with integral constant r yields

$$r+cU+\frac{k\mu U^3}{3}+k^3\tau U''=0 \quad ...(3.1.4)$$

In view of the homogenous balance principle, Equation (3.1.4) serves the value of present in Equation (2.2.4) as $n = 1$ for which the solution Equation (2.2.4) takes the form

$$U(\xi) = a_0 + a_1\Phi + b_1\Phi^{-1} \qquad ...(3.1.5)$$

where $\Phi = \varepsilon + (G'/G^2)$.

Equation (3.1.4) along with Equations (3.1.5) and (2.2.5) makes a polynomial in Φ. Equating like terms of this polynomial to zero gives a set of algebraic equations for a_0, a_1, b_1, c, k and r. Solving this system of equations by computer software Maple delivers the following outcomes:

Set 1: $$a_0 = -\frac{a_1(2s\rho - \mu)}{2\rho},\ b_1 = 0,\ r = 0,\ c = \mp\frac{a_1^3\eta\Delta\sqrt{-6\eta\tau}}{72\rho^3\tau},\ k = \pm\frac{a_1\sqrt{-6\eta\tau}}{6\rho\tau} \qquad ...(3.1.6)$$

where a_1 is an unknown constant.

Set 2: $$a_0 = -\frac{b_1(2s\rho - \mu)}{2(s^2\rho - s\mu + \sigma)},\ a_1 = 0,\ r = 0,\ c = \mp\frac{b_1^3\eta\Delta\sqrt{-6\eta\tau}}{72\tau(s^2\rho - s\mu + \sigma)^3},\ k = \pm\frac{b_1\sqrt{-6\eta\tau}}{6\tau(s^2\rho - s\mu + \sigma)} \qquad ...(3.1.7)$$

where b_1 is a free parameter.

Set 3: $$a_0 = -\frac{a_1}{2\rho}(2\varepsilon\rho - \mu),\ b_1 = \frac{a_1}{\rho}(\varepsilon^2\rho - \varepsilon\mu + \sigma),\ k = \pm\frac{a_1\sqrt{-6\eta\tau}}{6\rho\tau}$$

$$c = \mp\frac{a_1^3\eta\sqrt{-6\eta\tau}}{72\rho^3\tau}(12\varepsilon^2\rho^2 - 12\varepsilon\mu\rho + \mu^2 + 8\rho\sigma)$$

$$r = \pm\frac{a_1^4\eta\sqrt{-6\eta\tau}}{18\rho^3\tau}(2\varepsilon\rho\sigma - 3\varepsilon^2\mu\rho - \mu\sigma + \varepsilon\mu^2 + 2\varepsilon^3\rho^2) \qquad ...(3.1.8)$$

where a_1 is an arbitrary parameter.

Inserting the values appeared in Equations (3.1.6)-(3.1.8) into the solution (3.1.5) leaves the followings:

$$U_1(\xi) = \frac{a_1\mu}{2\rho} + a_1(G'/G^2) \qquad ...(3.1.9)$$

where $\xi = \mp\frac{a_1^3\eta\Delta\sqrt{-6\eta\tau}}{72\rho^3\tau}x \pm \frac{a_1\sqrt{-6\eta\tau}}{6\rho\tau}t$.

$$U_2(\xi) = \frac{b_1(2s\rho - \mu)}{2(s^2\rho - s\mu + \sigma)} + b_1(\varepsilon + G'/G^2)^{-1} \qquad ...(3.1.10)$$

where $\xi = \mp\frac{b_1^3\eta\Delta\sqrt{-6\eta\tau}}{72\tau(s^2\rho - s\mu + \sigma)^3}x \pm \frac{b_1\sqrt{-6\eta\tau}}{6\tau(s^2\rho - s\mu + \sigma)}t$.

$$U_3(\xi) = \frac{a_1\mu}{2\rho} + a_1(G'/G^2) + \frac{a_1}{\rho}(\varepsilon^2\rho - \varepsilon\mu + \sigma)(\varepsilon + G'/G^2)^{-1} \qquad ...(3.1.11)$$

where $\xi = \mp\frac{a_1^3\eta\sqrt{-6\eta\tau}}{72\rho^3\tau}(12\varepsilon^2\rho^2 - 12\varepsilon\mu\rho + \mu^2 + 8\rho\sigma)x \pm \frac{a_1\sqrt{-6\eta\tau}}{6\rho\tau}t$.

Eqs. (3.1.9)-(3.1.11) together with the results in (2.2.7) make available fifteen solutions to Eq. (3.1.1) as follows:

Solution Family 1:

$$U_1^1(\xi) = \frac{a_1\sqrt{\rho\sigma}\left(A\cos(\sqrt{\rho\sigma}\xi) + B\sin(\sqrt{\rho\sigma}\xi)\right)}{\sigma\left(B\cos(\sqrt{\rho\sigma}\xi) - A\sin(\sqrt{\rho\sigma}\xi)\right)} \qquad ...(3.1.12)$$

$$U_1^2(\xi) = \frac{a_1\sqrt{|\rho\sigma|}\left(A\sin h(2\sqrt{|\rho\sigma|}\xi) + A\cos h(2\sqrt{|\rho\sigma|}\xi) + B\right)}{\sigma\left(A\sin h(2\sqrt{|\rho\sigma|}\xi) + A\cos h(2\sqrt{|\rho\sigma|}\xi) - B\right)} \qquad ...(3.1.13)$$

$$U_1^3(\xi)=\frac{a_1A}{\rho(A\xi+B)} \quad ...(3.1.14)$$

$$U_1^4(\xi)=\frac{a_1\sqrt{\Delta}\left(A\cos h\left(\sqrt{\Delta}/2\xi\right)+B\sin h\left(\sqrt{\Delta}/2\xi\right)\right)}{2\rho\left(B\cos h\left(\sqrt{\Delta}/2\xi\right)+A\sin h\left(\sqrt{\Delta}/2\xi\right)\right)} \quad ...(3.1.15)$$

$$U_1^5(\xi)=-\frac{a_1\sqrt{-\Delta}\left(A\cos\left(\sqrt{-\Delta}/2\xi\right)+B\sin\left(\sqrt{-\Delta}/2\xi\right)\right)}{2\rho\left(B\cos\left(\sqrt{-\Delta}/2\xi\right)+A\sin\left(\sqrt{-\Delta}/2\xi\right)\right)} \quad ...(3.1.16)$$

where $\xi=\mp\frac{a_1^3\eta\Delta\sqrt{-6\eta\tau}}{72\rho^3\tau}x\pm\frac{a_1\sqrt{-6\eta\tau}}{6\rho\tau}t$.

Solution Family 2:

$$U_2^1(\xi)=-\frac{b_1s\rho}{(s^2\rho+\sigma)}+b_1\left(\varepsilon+\frac{\sqrt{\rho\sigma}\left(A\cos\left(\sqrt{\rho\sigma}\xi\right)+B\sin\left(\sqrt{\rho\sigma}\xi\right)\right)}{\sigma\left(B\cos\left(\sqrt{\rho\sigma}\xi\right)-A\sin\left(\sqrt{\rho\sigma}\xi\right)\right)}\right)^{-1} \quad ...(3.1.17)$$

$$U_2^2(\xi)=-\frac{b_1s\rho}{(s^2\rho+\sigma)}+b_1\left(\varepsilon-\frac{\sqrt{|\rho\sigma|}\left(A\sin h\left(2\sqrt{|\rho\sigma|}\xi\right)+A\cos h\left(2\sqrt{|\rho\sigma|}\xi\right)+B\right)}{\sigma\left(A\sin h\left(2\sqrt{|\rho\sigma|}\xi\right)+A\cos h\left(2\sqrt{|\rho\sigma|}\xi\right)-B\right)}\right)^{-1} \quad ...(3.1.18)$$

$$U_2^3(\xi)=-\frac{b_1s\rho}{(s^2\rho+\sigma)}+b_1\left(\varepsilon-\frac{A}{\rho(A\xi+B)}\right)^{-1} \quad ...(3.1.19)$$

$$U_2^4(\xi)=-\frac{b_1(2s\rho-\mu)}{2(s^2\rho-s\mu+\sigma)}+b_1\left(\varepsilon-\frac{\mu}{2\rho}-\frac{\sqrt{\Delta}\left(A\cos h\left(\sqrt{\Delta}/2\xi\right)+B\sin h\left(\sqrt{\Delta}/2\xi\right)\right)}{2\rho\left(B\cos h\left(\sqrt{\Delta}/2\xi\right)+A\sin h\left(\sqrt{\Delta}/2\xi\right)\right)}\right)^{-1} \quad ...(3.1.20)$$

$$U_2^5(\xi)=-\frac{b_1(2s\rho-\mu)}{2(s^2\rho-s\mu+\sigma)}+b_1\left(\varepsilon-\frac{\mu}{2\rho}-\frac{\sqrt{-\Delta}\left(A\cos\left(\sqrt{-\Delta}/2\xi\right)+B\sin\left(\sqrt{-\Delta}/2\xi\right)\right)}{2\rho\left(B\cos\left(\sqrt{-\Delta}/2\xi\right)+A\sin\left(\sqrt{-\Delta}/2\xi\right)\right)}\right)^{-1} \quad ...(3.1.21)$$

where $\xi=\mp\frac{b_1^3\eta\Delta\sqrt{-6\eta\tau}}{72\tau(s^2\rho-s\mu+\sigma)^3}x\pm\frac{b_1\sqrt{-6\eta\tau}}{6\tau(s^2\rho-s\mu+\sigma)}t$.

Solution Family 3:

$$U_3^1(\xi)=\frac{a_1\sqrt{\rho\sigma}\left(A\cos\left(\sqrt{\rho\sigma}\xi\right)+B\sin\left(\sqrt{\rho\sigma}\xi\right)\right)}{\sigma\left(B\cos\left(\sqrt{\rho\sigma}\xi\right)-A\sin\left(\sqrt{\rho\sigma}\xi\right)\right)}$$

$$+\frac{a_1}{\rho}(\varepsilon^2\rho+\sigma)\left(\varepsilon+\frac{\sqrt{\rho\sigma}\left(A\cos\left(\sqrt{\rho\sigma}\xi\right)+B\sin\left(\sqrt{\rho\sigma}\xi\right)\right)}{\sigma\left(B\cos\left(\sqrt{\rho\sigma}\xi\right)-A\sin\left(\sqrt{\rho\sigma}\xi\right)\right)}\right)^{-1} \quad ...(3.1.22)$$

$$U_3^2(\xi)=\frac{a_1\sqrt{|\rho\sigma|}\left(A\sin h\left(2\sqrt{|\rho\sigma|}\xi\right)+A\cos h\left(2\sqrt{|\rho\sigma|}\xi\right)+B\right)}{\sigma\left(A\sin h\left(2\sqrt{|\rho\sigma|}\xi\right)+A\cos h\left(2\sqrt{|\rho\sigma|}\xi\right)-B\right)}$$

$$+\frac{a_1}{\rho}(\varepsilon^2\rho+\sigma)\left(\varepsilon-\frac{\sqrt{|\rho\sigma|}\left(A\sin h\left(2\sqrt{|\rho\sigma|}\xi\right)+A\cos h\left(2\sqrt{|\rho\sigma|}\xi\right)+B\right)}{\sigma\left(A\sin h\left(2\sqrt{|\rho\sigma|}\xi\right)+A\cos h\left(2\sqrt{|\rho\sigma|}\xi\right)-B\right)}\right)^{-1} \quad ...(3.1.23)$$

$$U_3^3(\xi) = -\frac{a_1 A}{\rho(A\xi + B)} + \frac{a_1}{\rho}(\varepsilon^2\rho + \sigma)\left(\varepsilon - \frac{A}{\rho(A\xi + B)}\right)^{-1} \quad ...(3.1.24)$$

$$U_3^4(\xi) = \frac{a_1\sqrt{\Delta}\left(A\cos h\left(\sqrt{\Delta}/2\xi\right) + B\sin h\left(\sqrt{\Delta}/2\xi\right)\right)}{2\rho\left(B\cos h\left(\sqrt{\Delta}/2\xi\right) + A\sin h\left(\sqrt{\Delta}/2\xi\right)\right)}$$

$$+\frac{a_1}{\rho}(\varepsilon^2\rho - \varepsilon\mu + \sigma)\left(\varepsilon - \frac{\mu}{2\rho} - \frac{\sqrt{\Delta}\left(A\cos h\left(\sqrt{\Delta}/2\xi\right) + B\sin h\left(\sqrt{\Delta}/2\xi\right)\right)}{2\rho\left(B\cos h\left(\sqrt{\Delta}/2\xi\right) + A\sin h\left(\sqrt{\Delta}/2\xi\right)\right)}\right)^{-1} \quad ...(3.1.25)$$

$$U_3^5(\xi) = -\frac{a_1\sqrt{-\Delta}\left(A\cos\left(\sqrt{-\Delta}/2\xi\right) + B\sin\left(\sqrt{-\Delta}/2\xi\right)\right)}{2\rho\left(B\cos\left(\sqrt{-\Delta}/2\xi\right) + A\sin\left(\sqrt{-\Delta}/2\xi\right)\right)}$$

$$+\frac{a_1}{\rho}(\varepsilon^2\rho - \varepsilon\mu + \sigma)\left(\varepsilon - \frac{\mu}{2\rho} - \frac{\sqrt{-\Delta}\left(A\cos\left(\sqrt{-\Delta}/2\xi\right) + B\sin\left(\sqrt{-\Delta}/2\xi\right)\right)}{2\rho\left(B\cos\left(\sqrt{-\Delta}/2\xi\right) + A\sin\left(\sqrt{-\Delta}/2\xi\right)\right)}\right)^{-1} \quad ...(3.1.26)$$

where $\xi = \mp\frac{a_1^3\eta\sqrt{-6\eta\tau}}{72\rho^3\tau}\left(12\varepsilon^2\rho^2 - 12\varepsilon\mu\rho + \mu^2 + 8\rho\sigma\right)x \pm \frac{a_1\sqrt{-6\eta\tau}}{6\rho\tau}t$.

3.2. The Space-Time Fractional SRLW Equation

The space-time fractional SRLW equation is

$$D_t^{2\alpha}u + D_x^{2\alpha}u + uD_t^{\alpha}\left(D_t^{\alpha}u\right) + D_t^{\alpha}uD_x^{\alpha}u + D_t^{2\alpha}\left(D_x^{2\alpha}u\right) = 0 \quad ...(3.2.1)$$

The wave variable transformation

$$u(x, t) = U(\xi),\ \xi = kx + ct \quad ...(3.2.2)$$

reduces Equation (3.2.1) to the following fractional order nonlinear ordinary differential equation:

$$c^2U'' + k^2U'' + ckUU'' + ckU'^2 + c^2k^2U^{(iv)} = 0 \quad ...(3.2.3)$$

where U^1 denotes the α-order fractional derivative due to ξ. Integrate Equation (3.2.3) twice and the constants of integration are supposed to be zero left

$$\left(c^2 + k^2\right)U + \frac{ck}{2}U^2 + c^2k^2U'' = 0 \quad ...(3.2.4)$$

Due to the homogenous balance method, Equation (3.2.4) ensures the value of n present in Equation (2.2.4) to be $n = 2$ and the solution Equation (2.2.4) takes the form

$$U(\xi) = a_0 + a_1\Phi + a_2\Phi^2 + b_1\Phi^{-1} + b_2\Phi^{-2} \quad ...(3.2.5)$$

where $\Phi = \varepsilon + \left(G'/G^2\right)$.

Equation (3.2.4) with the help of Equations (3.2.5) and (2.2.5) provides a polynomial in Φ. Equating the coefficients of like terms of this polynomial to zero gives a system of algebraic equations for a_0, a_1, a_2, b_1, b_2, c and k. Solving this system of equations by computer software Maple brings the following results:

Set 1: $a_0 = \mp\frac{2c^2}{\sqrt{c^2\Delta - 1}}\left(2\rho\sigma + \mu^2 - 6\varepsilon\rho\sigma + 6\varepsilon^2\rho^2\right)$, $a_1 = \pm\frac{12c^2\rho}{\sqrt{c^2\Delta - 1}}(2\varepsilon\rho - \mu)$

$a_2 = \mp\frac{12c^2\rho^2}{\sqrt{c^2\Delta - 1}}$, $b_1 = 0$, $b_2 = 0$, $k = \mp\frac{c}{\sqrt{c^2\Delta - 1}}$ ...(3.2.6)

Set 2: $a_0 = \pm\frac{12c^2\rho}{\sqrt{-1 - c^2\Delta}}\left(\varepsilon^2\rho - \varepsilon\mu + \sigma\right)$, $a_1 = \mp\frac{12c^2\rho}{\sqrt{-1 - c^2\Delta}}(2\varepsilon\rho - \mu)$

$$a_2 = \pm\frac{12c^2\rho^2}{\sqrt{-1-c^2\Delta}},\ b_1 = 0,\ b_2 = 0,\ k = \mp\frac{c}{\sqrt{-1-c^2\Delta}} \qquad ...(3.2.7)$$

Set 3:

$$a_0 = \mp\frac{2c^2}{\sqrt{c^2\Delta-1}}\left(2\rho\sigma+\mu^2-6\varepsilon\rho\sigma+6\varepsilon^2\rho^2\right),\ a_1 = 0,\ a_2 = 0$$

$$b_1 = \pm\frac{12c^2}{\sqrt{c^2\Delta-1}}\left(2\varepsilon\rho\sigma-3\varepsilon^2\mu\rho-\mu\sigma+\varepsilon\mu^2+2\varepsilon^3\rho^2\right),\ k = \pm\frac{c}{\sqrt{c^2\Delta-1}}$$

$$b_2 = \mp\frac{12c^2}{\sqrt{c^2\Delta-1}}\left(\varepsilon^2\mu^2+\varepsilon^4\rho^2-2\varepsilon\mu\sigma+2\varepsilon^2\rho\sigma-2\varepsilon^3\mu\rho+\sigma^2\right) \qquad ...(3.2.8)$$

Set 4:

$$a_0 = \pm\frac{12c^2\rho}{\sqrt{-1-c^2\Delta}}\left(\varepsilon^2\rho-\varepsilon\mu+\sigma\right),\ a_1 = 0,\ a_2 = 0,\ k = \mp\frac{c}{\sqrt{-1-c^2\Delta}}$$

$$b_1 = \mp\frac{12c^2}{\sqrt{-1-c^2\Delta}}\left(2\varepsilon\rho\sigma-3\varepsilon^2\mu\rho-\mu\sigma+\varepsilon\mu^2+2\varepsilon^3\rho^2\right)$$

$$b_2 = \pm\frac{12c^2}{\sqrt{-1-c^2\Delta}}\left(\varepsilon^2\mu^2+\varepsilon^4\rho^2-2\varepsilon\mu\sigma+2\varepsilon^2\rho\sigma-2\varepsilon^3\mu\rho+\sigma^2\right) \qquad ...(3.2.9)$$

Inserting the values appeared in Equations (3.2.6)-(3.2.9) into the solution (3.2.5) leaves the followings:

$$U_1(\xi) = \mp\frac{2c^2}{\sqrt{c^2\Delta-1}}\left\{\left(2\rho\sigma+\mu^2-6\varepsilon\rho\sigma+\varepsilon\mu\rho+4\varepsilon^2\rho^2\right)-\rho\left(2\varepsilon\rho-\mu\right)\left(G'/G^2\right)+\rho^2\left(\varepsilon+G'/G^2\right)^2\right\} \qquad ...(3.2.10)$$

where $\xi = \mp\frac{c}{\sqrt{c^2\Delta-1}}x+ct$.

$$U_2(\xi) = \pm\frac{12c^2\rho}{\sqrt{-1-c^2\Delta}}\left\{\left(\sigma-\varepsilon^2\rho\right)-\left(2\varepsilon\rho-\mu\right)\left(G'/G^2\right)+\rho\left(\varepsilon+G'/G^2\right)^2\right\} \qquad ...(3.2.11)$$

where $\xi = \mp\frac{c}{\sqrt{-1-c^2\Delta}}x+ct$.

$$U_3(\xi) = \mp\frac{2c^2}{\sqrt{c^2\Delta-1}}\left\{\left(2\rho\sigma+\mu^2-6\varepsilon\rho\sigma+6\varepsilon^2\rho^2\right)-6\left(2\varepsilon\rho\sigma-3\varepsilon^2\mu\rho-\mu\sigma+\varepsilon\mu^2+2\varepsilon^3\rho^2\right)\left(\varepsilon+G'/G^2\right)^{-1}\right.$$

$$\left.+6\left(\varepsilon^2\mu^2+\varepsilon^4\rho^2-2\varepsilon\mu\sigma+2\varepsilon^2\rho\sigma-2\varepsilon^3\mu\rho+\sigma^2\right)\left(\varepsilon+G'/G^2\right)^{-2}\right\} \qquad ...(3.2.12)$$

where $\xi = \pm\frac{c}{\sqrt{c^2\Delta-1}}x+ct$.

$$U_4(\xi) = \pm\frac{12c^2}{\sqrt{-1-c^2\Delta}}\left\{\rho\left(\varepsilon^2\rho-\varepsilon\mu+\sigma\right)-\left(2\varepsilon\rho\sigma-3\varepsilon^2\mu\rho-\mu\sigma+\varepsilon\mu^2+2\varepsilon^3\rho^2\right)\left(\varepsilon+G'/G^2\right)^{-1}\right.$$

$$\left.+\left(\varepsilon^2\mu^2+\varepsilon^4\rho^2-2\varepsilon\mu\sigma+2\varepsilon^2\rho\sigma-2\varepsilon^3\mu\rho+\sigma^2\right)\left(\varepsilon+G'/G^2\right)^{-2}\right\} \qquad ...(3.2.13)$$

where $\xi = \mp\frac{c}{\sqrt{-1-c^2\Delta}}x+ct$.

Utilizing the results in (2.2.7) Equations (3.2.10)-(3.2.13) make available the following twenty solutions to Equation (3.2.1):

Solution Family 1:

$$U_1^1(\xi) = \mp\frac{2c^2}{\sqrt{c^2\Delta-1}}\left\{\left(2\rho\sigma-6\varepsilon\rho\sigma+4\varepsilon^2\rho^2\right)-\frac{2s\rho^2\sqrt{\rho\sigma}\left(A\cos\left(\sqrt{\rho\sigma}\xi\right)+B\sin\left(\sqrt{\rho\sigma}\xi\right)\right)}{\sigma\left(B\cos\left(\sqrt{\rho\sigma}\xi\right)-A\sin\left(\sqrt{\rho\sigma}\xi\right)\right)}\right.$$

$$\left.+\rho^2\left(\varepsilon+\frac{\sqrt{\rho\sigma}\left(A\cos\left(\sqrt{\rho\sigma}\xi\right)+B\sin\left(\sqrt{\rho\sigma}\xi\right)\right)}{\sigma\left(B\cos\left(\sqrt{\rho\sigma}\xi\right)-A\sin\left(\sqrt{\rho\sigma}\xi\right)\right)}\right)^2\right\} \qquad ...(3.2.14)$$

$$U_1^2(\xi)=\mp\frac{2c^2}{\sqrt{c^2\Delta-1}}\left\{\left(2\rho\sigma-6\varepsilon\rho\sigma+4\varepsilon^2\rho^2\right)+\frac{2s\rho^2\sqrt{|\rho\sigma|}\left(A\sin h\left(2\sqrt{|\rho\sigma|}\xi\right)+A\cos h\left(2\sqrt{|\rho\sigma|}\xi\right)+B\right)}{\sigma\left(A\sin h\left(2\sqrt{|\rho\sigma|}\xi\right)+A\cos h\left(\sqrt{|\rho\sigma|}\xi\right)-B\right)}\right.$$
$$\left.+\rho^2\left(\varepsilon-\frac{\sqrt{|\rho\sigma|}\left(A\sin h\left(2\sqrt{|\rho\sigma|}\xi\right)+A\cos h\left(2\sqrt{|\rho\sigma|}\xi\right)+B\right)}{\sigma\left(A\sin h\left(2\sqrt{|\rho\sigma|}\xi\right)+A\cos h\left(2\sqrt{|\rho\sigma|}\xi\right)-B\right)}\right)^2\right\}, \quad ...(3.2.15)$$

$$U_1^3(\xi)=\mp\frac{2c^2}{\sqrt{c^2\Delta-1}}\left\{\left(2\rho\sigma-6\varepsilon\rho\sigma+4\varepsilon^2\rho^2\right)+\frac{2s\rho^2A}{\rho(A\xi+B)}+\rho^2\left(\varepsilon-\frac{A}{\rho(A\xi+B)}\right)^2\right\}, \quad ...(3.2.16)$$

$$U_1^4(\xi)=\mp\frac{2c^2}{\sqrt{c^2\Delta-1}}\left\{\left(2\rho\sigma+\mu^2/2-6\varepsilon\rho\sigma+2\varepsilon\mu\rho+4\varepsilon^2\rho^2\right)+\frac{(2s\rho-\mu)\sqrt{\Delta}\left(A\cos h\left(\sqrt{\Delta}/2\xi\right)+B\sin h\left(\sqrt{\Delta}/2\xi\right)\right)}{2\left(B\cos h\left(\sqrt{\Delta}/2\xi\right)+A\sin h\left(\sqrt{\Delta}/2\xi\right)\right)}\right.$$
$$\left.+\left(\varepsilon\rho-\frac{\mu}{2}-\frac{\sqrt{\Delta}\left(A\cos h\left(\sqrt{\Delta}/2\xi\right)+B\sin h\left(\sqrt{\Delta}/2\xi\right)\right)}{2\left(B\cos h\left(\sqrt{\Delta}/2\xi\right)+A\sin h\left(\sqrt{\Delta}/2\xi\right)\right)}\right)^2\right\} \quad ...(3.2.17)$$

$$U_1^5(\xi)=\mp\frac{2c^2}{\sqrt{c^2\Delta-1}}\left\{\left(2\rho\sigma+\mu^2/2-6\varepsilon\rho\sigma+2\varepsilon\mu\rho+4\varepsilon^2\rho^2\right)+\frac{(2s\rho-\mu)\sqrt{-\Delta}\left(A\cos\left(\sqrt{-\Delta}/2\xi\right)+B\sin\left(\sqrt{-\Delta}/2\xi\right)\right)}{2\left(B\cos\left(\sqrt{-\Delta}/2\xi\right)+A\sin\left(\sqrt{-\Delta}/2\xi\right)\right)}\right.$$
$$\left.+\left(\varepsilon\rho-\frac{\mu}{2}-\frac{\sqrt{-\Delta}\left(A\cos\left(\sqrt{-\Delta}/2\xi\right)+B\sin\left(\sqrt{-\Delta}/2\xi\right)\right)}{2\left(B\cos\left(\sqrt{-\Delta}/2\xi\right)+A\sin\left(\sqrt{-\Delta}/2\xi\right)\right)}\right)^2\right\}, \quad ...(3.2.18)$$

where $\xi=\mp\frac{c}{\sqrt{c^2\Delta-1}}x+ct$.

Solution Family 2:

$$U_2^1(\xi)=\pm\frac{12c^2\rho}{\sqrt{-1-c^2\Delta}}\left\{\left(\sigma-\varepsilon^2\rho\right)-(2\varepsilon\rho-\mu)\left(\frac{\sqrt{\rho\sigma}\left(A\cos\left(\sqrt{\rho\sigma}\xi\right)+B\sin\left(\sqrt{\rho\sigma}\xi\right)\right)}{\sigma\left(B\cos\left(\sqrt{\rho\sigma}\xi\right)-A\sin\left(\sqrt{\rho\sigma}\xi\right)\right)}\right)\right.$$
$$\left.+\rho\left(\varepsilon+\frac{\sqrt{\rho\sigma}\left(A\cos\left(\sqrt{\rho\sigma}\xi\right)+B\sin\left(\sqrt{\rho\sigma}\xi\right)\right)}{\sigma\left(B\cos\left(\sqrt{\rho\sigma}\xi\right)-A\sin\left(\sqrt{\rho\sigma}\xi\right)\right)}\right)^2\right\}, \quad ...(3.2.19)$$

$$U_2^2(\xi)=\pm\frac{12c^2\rho}{\sqrt{-1-c^2\Delta}}\left\{\left(\sigma-\varepsilon^2\rho\right)+(2\varepsilon\rho-\mu)\left(\frac{\sqrt{|\rho\sigma|}\left(A\sin h\left(2\sqrt{|\rho\sigma|}\xi\right)+A\cos h\left(2\sqrt{|\rho\sigma|}\xi\right)+B\right)}{\sigma\left(A\sin h\left(2\sqrt{|\rho\sigma|}\xi\right)+A\cos h\left(\sqrt{|\rho\sigma|}\xi\right)-B\right)}\right)\right.$$
$$\left.+\rho\left(\varepsilon-\frac{\sqrt{|\rho\sigma|}\left(A\sin h\left(2\sqrt{|\rho\sigma|}\xi\right)+A\cos h\left(2\sqrt{|\rho\sigma|}\xi\right)+B\right)}{\sigma\left(A\sin h\left(2\sqrt{|\rho\sigma|}\xi\right)+A\cos h\left(2\sqrt{|\rho\sigma|}\xi\right)-B\right)}\right)^2\right\}, \quad ...(3.2.20)$$

$$U_2^3(\xi)=\pm\frac{12c^2\rho}{\sqrt{-1-c^2\Delta}}\left\{\left(\sigma-\varepsilon^2\rho\right)+(2\varepsilon\rho-\mu)\left(\frac{A}{\rho(A\xi+B)}\right)+\rho\left(\varepsilon+\frac{A}{\rho(A\xi+B)}\right)^2\right\}, \quad ...(3.2.21)$$

$$U_2^4(\xi)=\pm\frac{12c^2\rho}{\sqrt{-1-c^2\Delta}}\left\{\left(\sigma-\varepsilon^2\rho\right)+(2\varepsilon\rho-\mu)\left(\frac{\mu}{2\rho}+\frac{\sqrt{\Delta}\left(A\cos h\left(\sqrt{\Delta}/2\xi\right)+B\sin h\left(\sqrt{\Delta}/2\xi\right)\right)}{2\rho\left(B\cos h\left(\sqrt{\Delta}/2\xi\right)+A\sin h\left(\sqrt{\Delta}/2\xi\right)\right)}\right)\right.$$

$$+\rho\left(\varepsilon-\frac{\mu}{2\rho}-\frac{\sqrt{\Delta}\left(A\cos h\left(\sqrt{\Delta}/2\xi\right)+B\sin h\left(\sqrt{\Delta}/2\xi\right)\right)}{2\rho\left(B\cos h\left(\sqrt{\Delta}/2\xi\right)+A\sin h\left(\sqrt{\Delta}/2\xi\right)\right)}\right)^2\Bigg\}, \quad ...(3.2.22)$$

$$U_2^5(\xi)=\pm\frac{12c^2\rho}{\sqrt{-1-c^2\Delta}}\Bigg\{\left(\sigma-\varepsilon^2\rho\right)+\left(2\varepsilon\rho-\mu\right)\left(\frac{\mu}{2\rho}+\frac{\sqrt{-\Delta}\left(A\cos\left(\sqrt{-\Delta}/2\xi\right)+B\sin\left(\sqrt{-\Delta}/2\xi\right)\right)}{2\rho\left(B\cos\left(\sqrt{-\Delta}/2\xi\right)+A\sin\left(\sqrt{-\Delta}/2\xi\right)\right)}\right)$$

$$+\rho\left(\varepsilon-\frac{\mu}{2\rho}-\frac{\sqrt{-\Delta}\left(A\cos\left(\sqrt{-\Delta}/2\xi\right)+B\sin\left(\sqrt{-\Delta}/2\xi\right)\right)}{2\rho\left(B\cos\left(\sqrt{-\Delta}/2\xi\right)+A\sin\left(\sqrt{-\Delta}/2\xi\right)\right)}\right)^2\Bigg\}, \quad ...(3.2.23)$$

where $\xi=\mp\frac{c}{\sqrt{-1-c^2\Delta}}x+ct$.

Solution Family 3:

$$U_3^1(\xi)=\mp\frac{4c^2}{\sqrt{c^2\Delta-1}}\Big\{\left(\rho\sigma-3\varepsilon\rho\sigma+3\varepsilon^2\rho^2\right)-6\left(\varepsilon\rho\sigma+\varepsilon^3\rho^2\right)$$

$$\times\left(\varepsilon+\frac{\sqrt{\rho\sigma}\left(A\cos\left(\sqrt{\rho\sigma}\xi\right)+B\sin\left(\sqrt{\rho\sigma}\xi\right)\right)}{\sigma\left(B\cos\left(\sqrt{\rho\sigma}\xi\right)-A\sin\left(\sqrt{\rho\sigma}\xi\right)\right)}\right)^{-1}+3\left(\varepsilon^4\rho^2+2\varepsilon^2\rho\sigma+\sigma^2\right)$$

$$\times\left(\varepsilon+\frac{\sqrt{\rho\sigma}\left(A\cos\left(\sqrt{\rho\sigma}\xi\right)+B\sin\left(\sqrt{\rho\sigma}\xi\right)\right)}{\sigma\left(B\cos\left(\sqrt{\rho\sigma}\xi\right)-A\sin\left(\sqrt{\rho\sigma}\xi\right)\right)}\right)^{-2}\Bigg\}, \quad ...(3.2.24)$$

$$U_3^2(\xi)=\mp\frac{4c^2}{\sqrt{c^2\Delta-1}}\Big\{\left(\rho\sigma-3\varepsilon\rho\sigma+3\varepsilon^2\rho^2\right)-6\left(\varepsilon\rho\sigma+\varepsilon^3\rho^2\right)$$

$$\times\left(\varepsilon-\frac{\sqrt{|\rho\sigma|}\left(A\sin h\left(2\sqrt{|\rho\sigma|}\xi\right)+A\cos h\left(2\sqrt{|\rho\sigma|}\xi\right)+B\right)}{\sigma\left(A\sin h\left(2\sqrt{|\rho\sigma|}\xi\right)+A\cos h\left(2\sqrt{|\rho\sigma|}\xi\right)-B\right)}\right)^{-1}+3\left(\varepsilon^4\rho^2+2\varepsilon^2\rho\sigma+\sigma^2\right)$$

$$\times\left(\varepsilon-\frac{\sqrt{|\rho\sigma|}\left(A\sin h\left(2\sqrt{|\rho\sigma|}\xi\right)+A\cos h\left(2\sqrt{|\rho\sigma|}\xi\right)+B\right)}{\sigma\left(A\sin h\left(2\sqrt{|\rho\sigma|}\xi\right)+A\cos h\left(2\sqrt{|\rho\sigma|}\xi\right)-B\right)}\right)^{-2}\Bigg\}, \quad ...(3.2.25)$$

$$U_3^3(\xi)=\mp\frac{4c^2}{\sqrt{c^2\Delta-1}}\Bigg\{\left(\rho\sigma-3\varepsilon\rho\sigma+3\varepsilon^2\rho^2\right)-6\left(\varepsilon\rho\sigma+\varepsilon^3\rho^2\right)\left(\varepsilon-\frac{A}{\rho(A\xi+B)}\right)^{-1}$$

$$+3\left(\varepsilon^4\rho^2+2\varepsilon^2\rho\sigma+\sigma^2\right)\left(\varepsilon-\frac{A}{\rho(A\xi+B)}\right)^{-2}\Bigg\}, \quad ...(3.2.26)$$

$$U_3^4(\xi)=\mp\frac{2c^2}{\sqrt{c^2\Delta-1}}\Big\{\left(2\rho\sigma+\mu^2-6\varepsilon\rho\sigma+6\varepsilon^2\rho^2\right)$$

$$-6\left(2\varepsilon\rho\sigma-3\varepsilon^2\mu\rho-\mu\sigma+\varepsilon\mu^2+2\varepsilon^3\rho^2\right)$$

$$\times\left(\varepsilon-\frac{\mu}{2\rho}-\frac{\sqrt{\Delta}\left(A\cos h\left(\sqrt{\Delta}/2\xi\right)+B\sin h\left(\sqrt{\Delta}/2\xi\right)\right)}{2\rho\left(B\cos h\left(\sqrt{\Delta}/2\xi\right)+A\sin h\left(\sqrt{\Delta}/2\xi\right)\right)}\right)^{-1}$$

$$+6\left(\varepsilon^2\mu^2+\varepsilon^4\rho^2-2\varepsilon\mu\sigma+2\varepsilon^2\rho\sigma-2\varepsilon^3\mu\rho+\sigma^2\right)$$

$$\times\left(\varepsilon-\frac{\mu}{2\rho}-\frac{\sqrt{\Delta}\left(A\cosh\left(\sqrt{\Delta}/2\xi\right)+B\sinh\left(\sqrt{\Delta}/2\xi\right)\right)}{2\rho\left(B\cosh\left(\sqrt{\Delta}/2\xi\right)+A\sinh\left(\sqrt{\Delta}/2\xi\right)\right)}\right)^{-2}\Bigg\}, \quad ...(3.2.27)$$

$$U_3^5(\xi)=\mp\frac{2c^2}{\sqrt{c^2\Delta-1}}\Big\{\left(2\rho\sigma+\mu^2-6\varepsilon\rho\sigma+6\varepsilon^2\rho^2\right)$$

$$-6\left(2\varepsilon\rho\sigma-3\varepsilon^2\mu\rho-\mu\sigma+\varepsilon\mu^2+2\varepsilon^3\rho^2\right)$$

$$\times\left(\varepsilon-\frac{\mu}{2\rho}-\frac{\sqrt{-\Delta}\left(A\cos\left(\sqrt{-\Delta}/2\xi\right)+B\sin\left(\sqrt{-\Delta}/2\xi\right)\right)}{2\rho\left(B\cos\left(\sqrt{-\Delta}/2\xi\right)+A\sin\left(\sqrt{-\Delta}/2\xi\right)\right)}\right)^{-1}$$

$$+6\left(\varepsilon^2\mu^2+\varepsilon^4\rho^2-2\varepsilon\mu\sigma+2\varepsilon^2\rho\sigma-2\varepsilon^3\mu\rho+\sigma^2\right)$$

$$\times\left(\varepsilon-\frac{\mu}{2\rho}-\frac{\sqrt{-\Delta}\left(A\cos\left(\sqrt{-\Delta}/2\xi\right)+B\sin\left(\sqrt{-\Delta}/2\xi\right)\right)}{2\rho\left(B\cos\left(\sqrt{-\Delta}/2\xi\right)+A\sin\left(\sqrt{-\Delta}/2\xi\right)\right)}\right)^{-2}\Bigg\}, \quad ...(3.2.28)$$

where $\xi=\pm\frac{c}{\sqrt{c^2\Delta-1}}x+ct$.

Solution Family 4:

$$U_4^1(\xi)=\pm\frac{12c^2}{\sqrt{-1-c^2\Delta}}\Bigg\{\rho\left(\varepsilon^2\rho+\sigma\right)-2\varepsilon\rho\left(\sigma+\varepsilon^2\rho\right)\left(\varepsilon+\frac{\sqrt{\rho\sigma}\left(A\cos\left(\sqrt{\rho\sigma}\xi\right)+B\sin\left(\sqrt{\rho\sigma}\xi\right)\right)}{\sigma\left(B\cos\left(\sqrt{\rho\sigma}\xi\right)-A\sin\left(\sqrt{\rho\sigma}\xi\right)\right)}\right)^{-1}$$

$$+\left(\varepsilon^4\rho^2+2\varepsilon^2\rho\sigma+\sigma^2\right)\left(\varepsilon+\frac{\sqrt{\rho\sigma}\left(A\cos\left(\sqrt{\rho\sigma}\xi\right)+B\sin\left(\sqrt{\rho\sigma}\xi\right)\right)}{\sigma\left(B\cos\left(\sqrt{\rho\sigma}\xi\right)-A\sin\left(\sqrt{\rho\sigma}\xi\right)\right).}\right)^{-2}\Bigg\}, \quad ...(3.2.29)$$

$$U_4^2(\xi)=\pm\frac{12c^2}{\sqrt{-1-c^2\Delta}}\Big\{\rho\left(\varepsilon^2\rho+\sigma\right)-2\varepsilon\rho\left(\sigma+\varepsilon^2\rho\right)$$

$$\times\left(\varepsilon-\frac{\sqrt{|\rho\sigma|}\left(A\sinh\left(2\sqrt{|\rho\sigma|}\xi\right)+A\cosh\left(2\sqrt{|\rho\sigma|}\xi\right)+B\right)}{\sigma\left(A\sinh\left(2\sqrt{|\rho\sigma|}\xi\right)+A\cosh\left(\sqrt{|\rho\sigma|}\xi\right)-B\right)}\right)^{-1}+\left(\varepsilon^4\rho^2+2\varepsilon^2\rho\sigma+\sigma^2\right)$$

$$\times\left(\varepsilon-\frac{\sqrt{|\rho\sigma|}\left(A\sinh\left(2\sqrt{|\rho\sigma|}\xi\right)+A\cosh\left(2\sqrt{|\rho\sigma|}\xi\right)+B\right)}{\sigma\left(A\sinh\left(2\sqrt{|\rho\sigma|}\xi\right)+A\cosh\left(2\sqrt{|\rho\sigma|}\xi\right)-B\right)}\right)^{-2}\Bigg\}, \quad ...(3.2.30)$$

$$U_4^3(\xi)=\pm\frac{12c^2}{\sqrt{-1-c^2\Delta}}\Big\{\rho\left(\varepsilon^2\rho+\sigma\right)-2\varepsilon\rho\left(\sigma+\varepsilon^2\rho\right)\left(\varepsilon-\frac{A}{\rho(A\xi+B)}\right)^{-1}$$

$$+\left(\varepsilon^4\rho^2+2\varepsilon^2\rho\sigma+\sigma^2\right)\left(\varepsilon-\frac{A}{\rho(A\xi+B)}\right)^{-2}\Bigg\}, \quad ...(3.2.31)$$

$$U_4^4(\xi)=\pm\frac{12c^2}{\sqrt{-1-c^2\Delta}}\Big\{\rho\left(\varepsilon^2\rho-\varepsilon\mu+\sigma\right)-\left(2\varepsilon\rho\sigma-3\varepsilon^2\mu\rho-\mu\sigma+\varepsilon\mu^2+2\varepsilon^3\rho^2\right)$$

$$\times\left(\varepsilon-\frac{\mu}{2\rho}-\frac{\sqrt{\Delta}\left(A\cosh\left(\sqrt{\Delta}/2\xi\right)+B\sinh\left(\sqrt{\Delta}/2\xi\right)\right)}{2\rho\left(B\cosh\left(\sqrt{\Delta}/2\xi\right)+A\sinh\left(\sqrt{\Delta}/2\xi\right)\right)}\right)^{-1}$$

$$+\left(\varepsilon^2\mu^2+\varepsilon^4\rho^2-2\varepsilon\mu\sigma+2\varepsilon^2\rho\sigma-2\varepsilon^3\mu\rho+\sigma^2\right)$$

$$\times\left(\varepsilon-\frac{\mu}{2\rho}-\frac{\sqrt{\Delta}\left(A\cosh\left(\sqrt{\Delta}/2\xi\right)+B\sinh\left(\sqrt{\Delta}/2\xi\right)\right)}{2\rho\left(B\cosh\left(\sqrt{\Delta}/2\xi\right)+A\sinh\left(\sqrt{\Delta}/2\xi\right)\right)}\right)^{-2}\Bigg\}, \qquad ...(3.2.32)$$

$$U_4^5(\xi)=\pm\frac{12c^2}{\sqrt{-1-c^2\Delta}}\Big\{\rho\left(\varepsilon^2\rho-\varepsilon\mu+\sigma\right)-\left(2\varepsilon\rho\sigma-3\varepsilon^2\mu\rho-\mu\sigma+\varepsilon\mu^2+2\varepsilon^3\rho^2\right)$$

$$\times\left(\varepsilon-\frac{\mu}{2\rho}-\frac{\sqrt{-\Delta}\left(A\cos\left(\sqrt{-\Delta}/2\xi\right)+B\sin\left(\sqrt{-\Delta}/2\xi\right)\right)}{2\rho\left(B\cos\left(\sqrt{-\Delta}/2\xi\right)+A\sin\left(\sqrt{-\Delta}/2\xi\right)\right)}\right)^{-1}$$

$$+\left(\varepsilon^2\mu^2+\varepsilon^4\rho^2-2\varepsilon\mu\sigma+2\varepsilon^2\rho\sigma-2\varepsilon^3\mu\rho+\sigma^2\right)$$

$$\times\left(\varepsilon-\frac{\mu}{2\rho}-\frac{\sqrt{-\Delta}\left(A\cos\left(\sqrt{-\Delta}/2\xi\right)+B\sin\left(\sqrt{-\Delta}/2\xi\right)\right)}{2\rho\left(B\cos\left(\sqrt{-\Delta}/2\xi\right)+A\sin\left(\sqrt{-\Delta}/2\xi\right)\right)}\right)^{-2}\Bigg\}, \qquad ...(3.2.33)$$

where $\xi=\mp\frac{c}{\sqrt{-1-c^2\Delta}}x+ct$.

4. Conclusion

The core aim of this study was to make available further general and fresh closed form analytic solutions to the nonlinear space-time fractional mKdV equation and the nonlinear space-time fractional SRLW equation through the proposed modified fractional generalized (G'/G^2)-expansion method. The offered method has successfully presented attractive solutions to the suggested equations and shown its high performance. So far, we know the achieved solutions are not available in the literature and might create a milestone in research area. Therefore, it may be claimed that the modified fractional generalized (G'/G^2)-expansion method in deriving the closed form analytical solutions is simple, straightforward and productive. This method may be taken into account for further implementation to investigate any fractional order nonlinear evolution equations arising in various fields of science and engineering. The obtained solutions in terms of trigonometric function, hyperbolic function and rational function containing many free parameters are claimed to be fresh and further general which will take place in the literature.

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