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A New Model for Polyatomic Gases in an Electromagnetic Field

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Abstract

Significant progress has recently been made in the field of polyatomic gases, in particular by Professors T Ruggeri, M Sugiyama and collaborators. But so far it has not yet been seen how they interact with an electromagnetic field. This is realized in the present paper. As a first step, we consider here the case when the gas is described only by the Euler Equations and the electromagnetic field by Maxwell's Equations in materials. To find the field equations, a supplementary conservation law is imposed which is the entropy principle for the Euler Equations, while for Maxwell's Equations is the energy; this is useful because in this way the whole set of equations becomes a symmetric hyperbolic system as usual in Extended Thermodynamics. One of the results is a restriction on the law connecting the magnetic field in the empty space and the electric field in materials to the electromotive force and its dual: they are the gradients of a scalar function. Obviously, two Maxwell's equations are not evolutive (The Gauss magnetic and electric laws).

Keywords: *Polyatomic gases, Extended thermodynamics, Maxwell's Equations*

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1. Introduction

In this article, we try to put together the new knowledges available from Extended Thermodynamics (ET) (Born and Infeld, 1934; Donato and Ruggeri, 1972; Liu and M¨uller, 1972; Ruggeri, 1973; Liu and M¨uller, 1983; Amendt and Weitzner, 1985; Liu *et al.*, 1986; Anile and Pennisi, 1991; Carrisi, 2011; Boillat *et al.*, 1994; M¨uller and Ruggeri, 1998; Gibbons and Herdeiro, 2001; Boillat and Ruggeri, 2004; Carrisi *et al.*, 2004; Carrisi, 2013; Carrisi *et al.*, 2014; Carrisi and Pennisi, 2014; Carrisi *et al.*, 2015) (concerning field field equations that meet the hyperbolic requirement) and extensive literature on Maxwell's equations in matter. So far in the ET models, such equations contain the influence of the electromagnetic field only through Lorentz 's force (see, Amendt and Weitzner, 1985; Anile and Pennisi, 1991; Carrisi, 2011); likewise, in the articles on Maxwell's equations in matter, the latter are not coupled with the equations for matter and even less with those for polyatomic gases. Since there is to be expected that they affect each other, it is natural that they should be modeled with a single set of equations. It is also expected that, in the absence of an electromagnetic field, this set of equations must coincide with those already known of ET and that, in the absence of equations for the

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matter of ET, such equation set must coincide or contain as particular cases the known knowledge about Maxwell's equations in matter. This result is achieved in this article. Indeed, here we consider only the 5 moments model as ET component equations, but the extension to the case of several moments will be straightforward because almost all models in ET take as the benchmark of equilibrium just that with 5 moments.

The articles (see, Born and Infeld, 1934; Donato and Ruggeri, 1972; Ruggeri, 1973; Boillat *et al.*, 1994; Gibbond and Heedeiro, 2001; Boillat and Ruggeri, 2004) deserve a particular attention, so in Section 5 we will analyze the model presented here in the light of the knowledges presented in those articles. Let's now cite other articles from which we moved. To this regard we like to cite (Liu and Muller, 1972) where a very complicated theory was presented for magnetizable and polarizable fluids; for example, two new equations were presented for the magnetization and for the polarization. We will see here that this is not necessary because those equations are consequences of the remaining ones. But at that time Extended Thermodynamics was not fully established as today, so we have now more opportunities to take advantage of the new knowledge.

In (Carrisi *et al.*, 2004) we find an attempt to improve (Liu and Muller, 1972), by using an extended set of independent variables and of corresponding equations as usual in Extended Thermodynamics; but many ad hoc hypothesis were introduced and the results were not fully satisfactory. In (Strumia, 1992) this problems were considered but without using the whole set of Maxwell's equations and without considering them jointly with the field equations for the material because only a particular result was searched.

On the other hand, Maxwell's equations in the empty space are easier; so they have fully investigated from the present point of view as in Pennisi (1996) and Arima *et al.* (2012). The goal of the present article is to find the corresponding results for Maxwell's equations coupled with those of the material. To this end, we take advantage of the new knowledges appeared recently in literature for polyatomic gases such as, Pavi´c *et al.* (2013), Arima *et al.* (2012 and 2014), Carrisi *et al.* (2015), Ruggeri and Sugiyama (2015), Carrisi (2015), Carrisi and Pennisi (2016), Carrisi *et al.* (2016, 2017, 2019, 2020, 2021); Pennisi and Ruggeri (2017, 2020); Pennisi (2021); Ruggeri and Sugiyama (2021) but, as first step, we consider only the Euler equations for describing the contribute of the material. So the whole set of field equations here considered is

$$\partial_t F + \partial_k F^k = 0 \text{ (Mass Conservation)} \tag{1}$$

$$\partial_t F^i + \partial_k F^{ki} = -q \in^i \text{ (Momentum Conservation)}$$

$$\partial_t G^{ll} + \partial_k G^{kll} = -2q \in^i \upsilon_i \text{ (Energy Conservation)}$$

$$\partial_t q + \partial_k j_k = 0 \text{ (Charge Conservation)}$$

$$\partial_t B^i + \partial_k \left(- \in^{kij} E_j\right) = 0 \text{ (Faraday's Conservation)}$$

$$\partial_t D^i + \partial_k \left(\in^{kij} H_j\right) = -j^i \text{ (Ampere-Maxwell's Law)}$$

$$\partial_k B^k = 0 \text{ (Gauss Magenetic law)}$$

$$\partial_k D^k = q \text{ (Gauss Electric Law)}$$

Here the independent variables are the mass density F, the momentum density F^i, the energy density G^{ll}, the magnetic field in the empty space B^i (or magnetic induction), the electric field in materials D^i (or electric induction) and the free charge density q.

The other quantities are constitutive functions of the independent variables; they are the flux of mass F^k, the momentum flux F^{ki}, the energy flux G^{kll}, the free electric current j^i, the electric field in the empty space E^i and the magnetic field in materials H^i. Moreover, $\in^{kij}$ is the Levi-Civita symbol. The right hand sides of $(1)_{2,3}$ are due to the presence of the Lorentz force, with

$$- \in^i = E^i + \left(\vec{\upsilon} \wedge \vec{B}\right)^i$$

Obviously, Equation $(1)_4$ is a consequence of $(1)_{6,8}$ but we prefer to retain it for the following reason:

The derivative with respect to x_i of $(1)_5$ is $\partial_t(\partial_i B^i)=0$ so that it will suffice to impose Equation $(1)_7$ on the initial manifold and, as a consequence, it will be satisfied also outside it. Similarly, the derivative with respect to x_i of $(1)_6$ gives $\partial_t(\partial_i D^i - q)=0$ which is now a consequence of Equation $(1)_4$ so that it will suffice to impose Equation $(1)_8$ on the initial manifold.

There is no need to introduce an equation for the polarization P^i and the magnetization M^i because they are already expressed in terms of the above independent variables through the definitions

$$P^i = D^i - \in^{ij} E_j,\ M^i = (\mu^{-1})^{ij} B_j - H^i \tag{2}$$

with $\in^{ij}$ and μ^{ij} invertible matrices.

Similarly, we can define the total charge density q_T and the total current density j_T^i from

$$\partial_k\left(\in^{kj} E_j\right) = qT,\ -\partial_t\left(\in^{ij} E_j\right) + \partial_k\left[\in^{ikj} (\mu^{-1})^{ja} B_a\right] = j_T^i \tag{3}$$

After that, from Equations (2) it follows

$$\partial_t P^i + \partial_k\left(\in^{ikj} M_j\right) = j_T^i - j^i,$$

$$-\partial_k P^k = qT - q$$

and these are not new balance equations but simply the definitions of the non free charge density $qT - q$ and of the non free electric current $J_T^i - j^i$. In the case without polarization and magnetization all the charges and all the currents are free. Moreover, from $P^i = 0$, $M^i = 0$ and from Equations (2) we deduce

$$D^i = \in^{ij} E_j, \quad H^i = (\mu^{-1})^{ia} B_a \tag{4}$$

and the Maxwell's equations $(1)_{5-8}$ become

$$\partial_t B^i + \partial_k\left(-\in^{kij} E_j\right) = 0, \qquad \partial_t\left(\in^{ij} E_j\right) + \partial_k\left[\in^{kij} \left(\mu^{-1}\right)^{ja} B_a\right] = -j^i$$

$$\partial_k B^k = 0, \qquad \partial_k\left(\in^{ka} E_a\right) = q$$

These are called the Maxwell's equations in homogeneous and isotropic media. Now we see the meaning of the definition (3) compared with Equations $(1)_{6,8}$: q_T and j_T are the charge and the current density we would have if the media was homogeneous and isotropic.

If $\in^{ij} = \in_0 \delta^{ij}$, $\mu^{ij} = \mu_0 \delta^{ij}$ with $\in_0$ (electric permittivity) and μ_0 (magnetic permeability) constants, the Maxwell's equations become

$$\partial_t B^i + \partial_k\left(-\in^{kij} E_j\right) = 0, \qquad \partial_t(E^i) - \frac{1}{\in_0 \mu_o}\partial_k\left(\in^{ikj} B_j\right) = \frac{j^i}{\in_0}$$

$$\partial_k B^k = 0, \qquad \partial_k\left(E^k\right) = \frac{q}{\in_0}$$

Moreover, Maxwell discovered, by using the values of $\in_0$ and μ_0 known from experiments in the empty space, that the quantity $\frac{1}{\sqrt{\in_0 \mu_0}}$ in the empty space is exactly equal to the speed of light c. In this case the above equations become the Maxwell's equations in empty space. In the more general case, $\frac{1}{\sqrt{\in_0 \mu_0}}$ can be considered as the speed of light in the material (Obviously, it is less than c). From these considerations it is evident that the above Equations (1) are the most general and we can study them.

In Section 2 we will impose on Equations $(1)_{1\text{-}8}$ the relativity principle and the existence of a supplementary conservation law. One of the results is a restriction on the law connecting the magnetic field in the empty space and the electric field in materials to the electromotive force $\vec{\in}$ and its dual $\vec{X}$ We will find that a scalar function h' exists such that $\vec{B}$ and $\vec{D}$ are derivatives of h' with respect to two lagrange multipliers related with $\vec{\in}$ and $\vec{X}$.

More precisely, since Equations $(1)_{7,8}$ are evolutive constraints, we will apply to the present system the methodology already known in literature for this case, such as, Strumia (1988), Boillat (1994), Pennisi (1997). In Section 3, by using the methodology of Ruggeri and Strumia (1981) and Ruggeri (1989), this will transform our system in the symmetric hyperbolic form with all its consequent good mathematical and physical properties. In Section 4 we will study the wave equations for our field equations, and in Section 5 we will compare our results with others already known in literature.

2. Existence of a Supplementary Conservation Law

First of all, it is useful to perform the following change of variables from (E^i, H^i) to ($\in^i$, X^i):

$$-\in^i = E^i + \left(\vec{v}\wedge\vec{B}\right)^i, \qquad X^i = H^i + \left(\vec{v}\wedge\vec{D}\right)^i \tag{5}$$

because $-\in^i$ is proportional to the Lorentz force and can be called the electromotive force; consequently, it doesn't depend on the observer. For this reason it is preferable to take $\in^i$ as variable instead of E^i. This fact justifies the choice $(5)_1$. As a matter of parallelism it is preferable to take $X^i = -H^i + \left(\vec{v}\wedge\vec{D}\right)^i$ as variable instead of H^i and this justifies the choice $(5)_2$. Moreover, we know that, under a transformation from a reference frame to another one, moving with respect to the previous one of a translational motion with velocity $\vec{v}$, B^i and D^i transform as pseudo-vectors (that is, as vectors but only under the orthonormal transformations which preserve the orientation). Regarding F, F^i, G^{ll} and the fluxes F^{ki}, G^{kll} we use the decomposition in Arima *et al*. (2012), that is

$F = F$, $F^i = Fv^i$, $G^{ll} = Fv^2 + m^{ll}$, $F^{ki} = Fv^k v^i + M^{ki}$, $G^{kll} = Fv^2 v^k + 2\, v_p M^{kp} + m^{ll} v^k + m^{kll}$,

where F, m^{ll}, M^{ki} and m^{kll} don't depend on the velocity.

Now we are ready to introduce the supplementary conservation law which must old for all the solutions of the field equations and it is

$$\partial_t h + \partial_k h^k = \sigma$$

Without the presence of the electromagnetic field and the further condition $\sigma \geq 0$, this is the entropy principle; in the presence of the electromagnetic field we don' t demand a physical meaning of this condition, except for the fact that it leads to a symmetric hyperbolic system. By using Liu's theorem (Liu, 1972), it is equivalent to assume the existence of Lagrange multipliers μ, λ_i, λ_{ll}, ϑ, μ_i, v_i such that

$$dh = \mu dF + \lambda_i dF^i + \lambda_{ll} dG^{ll} + \vartheta dq + \mu_i dB^i + v_i dD^i \tag{6}$$

$$dh^k = \mu dF^k + \lambda_i dF^{ki} + \lambda_{ll} dG^{kll} + \vartheta dj^k - \mu_i \in^{kij} dE_j + v_i \in^{kij} dH_j + \psi dB^k + \eta dD^k$$

$$\sigma = -v_i j^i + q\left(\lambda_i \in^i + 2\lambda_{ll} v_i \in^i + \eta\right)$$

where the scalar functions ψ and η are present to take into account the evolutive constraints $(1)_{7,8}$. Now Equation $(6)_1$ in the independent variables F, v^i, E, B^i, D^i and by assuming that h doesn't depend on v^i, gives

$$\mu = \hat{\mu} + \lambda_{ll} v^2, \; \lambda_i = -2\lambda_{ll} v_i, \; \lambda_{ll} = \frac{\partial h}{\partial m_{ll}}, \; \hat{\mu} = \frac{\partial h}{\partial F}, \tag{7}$$

$$\vartheta = \frac{\partial h}{\partial q}, \; \mu_i = \frac{\partial h}{\partial B_i}, v_i = \frac{\partial h}{\partial D_i}$$

Since h is a scalar function not depending on v^i, so we may deduce that also $\hat{\mu}$, λ_{ll}, ϑ, μ_i and v_i don't depend on v_i.

Similarly, by imposing that $\frac{\partial\left(h^k - hv^k\right)}{\partial v^i} = 0$ we obtain

$$0 = \hat{\mu}F\delta^{ik} + \lambda_{ll}\left(2M^{ki} + m^{ll}\delta^{ik}\right) - h\delta^{ik} + \vartheta\frac{\partial j^k}{\partial v_i} - \mu_r \in^{krs} \frac{\partial E_s}{\partial v_i} + v_r \in^{krs} \frac{\partial H_s}{\partial v_i}$$

By substituting here E_s and H_s from (5), this relation becomes

$$0 = \hat{\mu}F\delta^{ik} + \lambda_{ll}\left(2M^{ki} + m^{ll}\delta^{ik}\right) - h\delta^{ik} + \vartheta\frac{\partial j^k}{\partial v_i} - \mu_i B^k - v^i D^k + \left(\mu_r B^r + v_r D^r\right)\delta^{ik},$$

where we have used the identity $\in^{krs}\in_{sia} = 2\delta_i^{[k}\,\delta_a^{r]} = 2\delta_{[i}^{k}\,\delta_{a]}^{r}$. This result shows that $\frac{\partial j^k}{\partial v_i}$ does not depend on v_i and it seems obviuos to take $j^k = qv^k$. So the last relation, by using (7), becomes

$$0 = \left(F\frac{\partial h}{\partial F} + m^{ll}\frac{\partial h}{\partial m^{ll}} + q\frac{\partial h}{\partial q} + B_r\frac{\partial h}{\partial B_r} + D^r\frac{\partial h}{\partial D_r} - h\right)\delta^{ik} + 2\frac{\partial h}{\partial m^{ll}}M^{ik} - \mu^i B^k - v^i D^k \tag{8}$$

which can be used to deduce M^{ik}. After that, $(6)_{1,2}$ imply that

$$\frac{\partial\left(h^k - hv^k\right)}{\partial B_i} = \lambda_{ll}\frac{\partial m^{kll}}{\partial B_i} + \mu_r \in^{krj} \frac{\partial \in_j}{\partial B_i} - v_r \in^{krj} \frac{\partial x_j}{\partial B_i} + \left(\psi - \mu_r v^r\right)\delta^{ik},$$

$$\frac{\partial\left(h^k - hv^k\right)}{\partial D_i} = \lambda_{ll}\frac{\partial m^{kll}}{\partial D_i} + \mu_r \in^{krj} \frac{\partial \in_j}{\partial D_i} - v_r \in^{krj} \frac{\partial x_j}{\partial D_i} + \left(\eta - v_r v^r\right)\delta^{ik}$$

where we have used $(7)_2$, (5) and the above mentioned identity. Since these expressions must not depend on the velocity, it follows that $\psi = \mu_r v^r$, $\eta = v_r v^r$ except for additional terms not depending on v_i which we assume, for the sake of simplicity, to be zero. As a consequence of these results, we see that $(6)_3$ becomes $\sigma = 0$.

Aiming to obtain a symmetric system according to the ideas of [49], we introduce now the 4-potentials

$$h' = -h + \mu F + \lambda_i F^i + \lambda_{ll}G^{ll} + \vartheta q + \mu_i B^i + v_i D^i,$$

$$h'^k = -h^k + \mu F^k + \lambda_i F^{ki} + \lambda_{ll}G^{ll} + \vartheta j^k - \mu_i \in^{kij} E_j + v_i \in^{kij} H_j + \psi B^k + \eta D^k \tag{9}$$

In this way Equations $(6)_{1,2}$ become

$$dh' = Fd\mu + F^i d\lambda_i + G^{ll}d\lambda_{ll} + qd\vartheta + B^i d\mu_i + D^i dv_i,$$

$$dh'^k = F^k d\mu + F^{ki}d\lambda_i + G^{kll}d\lambda_{ll} + j^k d\vartheta - \in^{kij} E_j d\mu_i + \in^{kij} H_j dv_i + B^k d\psi + D^k d\eta \tag{10}$$

By using $(7)_{1,2}$ we see that h depends on μ and λ_i only by means of $\hat{\mu} = \mu - \frac{\lambda_i\lambda^i}{4\lambda_{ll}}$. By using $(9)_1$, we see that also h' has this property and $(10)_1$ becomes

$$dh' = Fd\hat{\mu} + m^{ll}d\lambda_{ll} + qd\vartheta + B^i d\mu_i + D^i dv_i \tag{11}$$

From Equations $(10)_2$ and (11) we obtain also

$$d\left(h'^k + h'\frac{\lambda^k}{2\lambda_{ll}}\right) = \left(h'\delta^{ki} + 2\lambda_{ll}M^{ki} - B^k\mu^i - D^k v^i\right)d\left(\frac{\lambda_i}{2\lambda_{ll}}\right) + m^{kll}d\lambda_{ll}$$

$$+\in^{kij}\in_j\ d\mu_i - \in^{kij}\ X_j dv_i = m^{kll} d\lambda_{ll} + \in^{kij}\in_j\ d\mu_i - \in^{kij}\ X_j dv_i \tag{12}$$

where we have used also $\psi = \mu_r v^r$, $\eta = v_r v^r$ and Equations (5), (7), (8). In particular, this last one becomes

$$h'\delta^{ki} + 2\lambda_{ll} M^{ki} - B^k \mu^i - D^k v^i = 0 \tag{13}$$

which defines M^{ki}.

So, up to now, we have found h' and $\hat{h}'^k = h'^k + h'\dfrac{\lambda^k}{2\lambda_{ll}}$ which depends on the scalars $\hat{\mu}, \lambda_{ll}, \vartheta$ and on the vectors μ_i, v_i. For the Representation Theorems (Pennisi and Trovato, 1989; and Pennisi, 1998), we have then

$$\hat{h}'^k = h_1\mu^k + h_2 v^k + h_3 \in^{krs}\ \mu_r v_s \tag{14}$$

and h', h_i are functions of $\hat{\mu}, \lambda_{ll}, \vartheta, Y_{11} = \mu_i\mu^i, Y_{12} = \mu_i v^i, Y_{22} = v_i v^i$. After that, Equation (12) becomes

$$\begin{aligned}
m^{kll} &= \frac{\partial h_1}{\partial \lambda_{ll}}\mu^k + \frac{\partial h_2}{\partial \lambda_{ll}} v^k + \frac{\partial h_3}{\partial \lambda_{ll}} \in^{krs}\ \mu_r v_s, \\
\in^{kij}\in_j &= h_1\delta^{ki} + \mu^k\left(2\frac{\partial h_1}{\partial Y_{11}}\mu^i + \frac{\partial h_1}{\partial Y_{12}}v^i\right) + v^k\left(2\frac{\partial h_2}{\partial Y_{11}}\mu^i + \frac{\partial h_2}{\partial Y_{12}}v^i\right) \\
&\quad + \in^{krs}\ \mu_r v_s\left(2\frac{\partial h_3}{\partial Y_{11}}\mu^i + \frac{\partial h_3}{\partial Y_{12}}v^i\right) + h_3 \in^{kis}\ v_s \\
\in^{kij}\ X_j &= h_2\delta^{ki} + \mu^k\left(\frac{\partial h_1}{\partial Y_{12}}\mu^i + 2\frac{\partial h_1}{\partial Y_{22}}v^i\right) + v^k\left(\frac{\partial h_2}{\partial Y_{12}}\mu^i + 2\frac{\partial h_2}{\partial Y_{22}}v^i\right) \\
&\quad + \in^{krs}\ \mu_r v_s\left(\frac{\partial h_3}{\partial Y_{12}}\mu^i + 2\frac{\partial h_3}{\partial Y_{22}}v^i\right) + h_3 \in^{kri}\ \mu_r
\end{aligned} \tag{15}$$

In the reference frame with $\mu^i \equiv (\mu^1, 0, 0)$, $v_i \equiv (v^1, v^2, 0)$, the components 33 of $(15)_{2,3}$ give $h_1 = 0$, $h_2 = 0$. The components 23 and 13 of their symmetric parts give $\dfrac{\partial h_3}{\partial Y_{12}} = 0$, $\dfrac{\partial h_3}{\partial Y_{22}} = 0$, $\dfrac{\partial h_3}{\partial Y_{11}} = 0$. Then h_3 depends only on λ_{ll} (From Equation (12) it cannot depend on the other scalars). So Equation (15) now simplifies in

$$m^{kll} = \frac{\partial h_3}{\partial \lambda_{ll}} \in^{krs}\ \mu_r v_s,\ \in_j = h_3 v_j,\ X_j = h_3\mu_j \tag{16}$$

This result gives physical meaning to the Lagrange multipliers v_j and μ_j; they are parallel to $\in_j$ and X_j; they can be also equal to them if $h_3 = 1$.

We see also that, from (11) it follows:

$$F = \frac{\partial h'}{\partial \hat{\mu}}, m^{ll} = \frac{\partial h'}{\partial \lambda_{ll}}, q = \frac{\partial h'}{\partial \vartheta}, B^i = \frac{\partial h'}{\partial \mu_i}, D^i = \frac{\partial h'}{\partial v_i} \tag{17}$$

The last two of these show the only restriction given by the existence of a supplementary conservation law: The functions relating B^i and D^i to μ_i and v_i are not arbitrary but gradients of a scalar function h' with respect to μ_i and v_i. Moreover, the material objectivity principle is satisfied.

The final result for h'^k is

$$h'^k = h'\frac{-\lambda^k}{2\lambda_{ll}} + h_3(\lambda_{ll}) \in^{krs}\ \mu_r v_s \tag{18}$$

Coming back to Equations (10) we may rewrite them using the compact notation λ_A to denote all the Lagrange multipliers; so we obtain

$$F^A = \frac{\partial h'}{\partial \lambda_A}, \; \frac{\partial h'^k}{\partial \lambda_A} = F^{kA} + B^k \frac{\partial \psi}{\partial \lambda_A} + D^k \frac{\partial \eta}{\partial \lambda_A}, \tag{19}$$

i.e., $$F = \frac{\partial h'}{\partial \hat{\mu}}, \; F^i = \frac{\partial h'}{\partial \hat{\mu}} \frac{-\lambda^i}{2\lambda_{ll}}, \; G^{ll} = \frac{\partial h'}{\partial \hat{\mu}} \frac{\lambda_r \lambda^r}{4(\lambda_{ll})^2} + \frac{\partial h'}{\partial \lambda_{ll}}, \; q = \frac{\partial h'}{\partial \vartheta}, \; B^i = \frac{\partial h'}{\partial \mu_i}, \; D^i = \frac{\partial h'}{\partial \nu_i} \tag{20}$$

$$F^k = \frac{\partial h'}{\partial \hat{\mu}} \frac{-\lambda^k}{2\lambda_{ll}}, \; F^{ki} = \frac{\partial h'}{\partial \hat{\mu}} \frac{\lambda^k \lambda^i}{4(\lambda_{ll})^2} - \frac{h'}{2\lambda_{ll}} \delta^{ki} + B_k \frac{\mu^i}{2\lambda_{ll}} + D^k \frac{\nu^i}{2\lambda_{ll}},$$

$$G^{kll} = G^{ll} \frac{-\lambda^k}{2\lambda_{ll}} + h' \frac{\lambda^k}{2(\lambda_{ll})^2} + \frac{\partial h_3}{\partial \lambda_{ll}} \in^{krs} \mu_r \nu_s - B^k \frac{\mu_r \lambda^r}{(2\lambda_{ll})^2} - D^k \frac{\nu_r \lambda^r}{2(\lambda_{ll})^2},$$

$$j^k = q \frac{-\lambda^k}{2\lambda_{ll}}, -\in^{kij} E_j = 2B^{[i} \frac{-\lambda^{k]}}{2\lambda_{ll}} + h_3(\lambda_{ll}) \in^{kis} \nu_s, \; \in^{kij} H_j = 2D^{[i} \frac{-\lambda^{k]}}{2\lambda_{ll}} + h_3(\lambda_{ll}) \in^{ksi} \mu_s,$$

where we have used

$$\hat{\mu} = \mu - \frac{\lambda_r \lambda^r}{4\lambda_{ll}}, \; \psi = \mu_r \frac{-\lambda^r}{2\lambda_{ll}}, \; \eta = \nu_r \frac{-\lambda^r}{2\lambda_{ll}}$$

We note that $(20)_{11,12}$, thanks to (5), confirm the above defined $(16)_{2,3}$.

3. The Hyperbolicity Requirement

Let us use Equations (19) to obtain F^A and F^{kA}; after that, we see that the field Equations $(1)_{1\text{-}6}$ become

$$\frac{\partial^2 h'}{\partial \lambda_A \partial \lambda_B} \partial_t \lambda_B + \left(\frac{\partial^2 h'^k}{\partial \lambda_A \partial \lambda_B} - B^k \frac{\partial^2 \psi}{\partial \lambda_A \partial \lambda_B} - D^k \frac{\partial^2 \eta}{\partial \lambda_A \partial \lambda_B} \right) \partial_k \lambda_B - \frac{\partial^2 \psi}{\partial \lambda_A} \partial_k B^k - \frac{\partial^2 \eta}{\partial \lambda_A} \partial_k D^k = p^A$$

If we add to this Equation $(1)_7$ multiplied by $\frac{\partial \psi}{\partial \lambda_A}$ and $(1)_8$ multiplied by $\frac{\partial \eta}{\partial \lambda_A}$, it becomes

$$\frac{\partial^2 h'}{\partial \lambda_A \partial \lambda_B} \partial_t \lambda_B + \left(\frac{\partial^2 h'^k}{\partial \lambda_A \partial \lambda_B} - B^k \frac{\partial^2 \psi}{\partial \lambda_A \partial \lambda_B} - D^k \frac{\partial^2 \eta}{\partial \lambda_A \partial \lambda_B} \right) \partial_k \lambda_B = p^A + q \frac{\partial \eta}{\partial \lambda_A} \tag{21}$$

Because the matrixes coefficients of $\partial_t \lambda_B$ and $\partial_k \lambda_B$ are symmetrix with respect to the multi-index A, B, for the hyperbolicity of the system (21) it suffices that the matrix $\frac{\partial_2 h'}{\partial \lambda_A \partial \lambda_B}$ is negative defined (Boillat and Ruggeri, 1997; Muller and Ruggeri, 1998), i.e., that h' is a concave function of the variables λ_A. In other words, we add to Equations $(1)_{1\text{-}6}$ a linear combination of $(1)_{7,8}$ and, after that, we leave out $(1)_{7,8}$; this can be done because they are now consequences of (21). In fact, by writing this new set explicitly, it reads

$$\partial_t F + \partial_k F^k = 0 \tag{22}$$

$$\partial_t F + \partial_k F^k - \frac{\mu^i}{2\lambda_{ll}} \partial_k B^k - \frac{\nu^i}{2\lambda_{ll}} \partial_k D^k = -q \frac{\nu^i}{2\lambda_{ll}} + q \in^i,$$

$$\partial_t G^{ll} + \partial_k G^{kll} + \frac{\lambda_r \mu^r}{2(\lambda_{ll})^2} \partial_k B^k + \frac{\lambda_r \nu^r}{2(\lambda_{ll})^2} \partial_k D^k = \frac{\lambda_r \nu^r}{2(\lambda_{ll})^2} q - 2q \in^i \frac{\lambda_i}{2\lambda_{ll}},$$

$$\partial_t q + \partial_{kj}^{k} = 0,$$

$$\partial_t B^i + \partial_k(-\in^{kij} E_j) - \frac{\lambda^i}{2\lambda_{ll}}\partial_k B^k = 0,$$

$$\partial_t D^i + \partial_k(\in^{kij} H_j) - \frac{\lambda^i}{2\lambda_{ll}}\partial_k D^k = 0,$$

where we have used $j^i = \frac{\lambda^i}{2\lambda_{ll}} q.$ Now, the derivatives with respect to x_i of $(22)_{5,6}$ are

$$\partial_t(\partial_k B^k) + \partial_i\left[\frac{-\lambda^i}{2\lambda_{ll}}\left(\partial_k B^k\right)\right] = 0, \quad \partial_t(\partial_k D^k - q) + \partial_i\left[\frac{-\lambda^i}{2\lambda_{ll}}\left(\partial_k D^k - q\right)\right] = 0 \tag{23}$$

The first one of these equations is obtained with the following passages:

$$\partial_t(\partial_k B^k) = \partial_i\left(\in^{kij} \partial_k E_j + \frac{\lambda^i}{2\lambda_{ll}}\partial_k B^k\right) = \partial_i\left[\frac{\lambda^i}{2\lambda_{ll}}\partial_k B^k\right]$$

where in the first passage we have substituted $\partial_t B^i$ from $(22)_5$ and in the second passage we have used the identity $\in^{kij} \partial_{ik} E_j = 0$. Similarly, we have

$$\partial_t(\partial_k D^k - q) = \partial_i\left(-\in^{kij} \partial_k H_j + \frac{\lambda^i}{2\lambda_{ll}}\partial_k D^k + \frac{-\lambda^i}{2\lambda_{ll}} q - j^i\right) + \partial_k j^k = \partial_i\left[\frac{\lambda^i}{2\lambda_{ll}}\left(\partial_k D^k - q\right)\right]$$

where in the first passage we have substituted $\partial_t D^i$ from $(22)_6$ and $\partial_t q$ from $(22)_4$ while in the second passage we have used the identity $\in^{kij} \partial_{ik} H_j = 0$. Moreover, we have used again $j^i = \frac{\lambda}{2\lambda_{ll}} q.$ The result (23) shows that it suffices to impose $(1)_{7,8}$ only in the initial manifold and, after that, they will be satisfied also outside of it as a conequence of $(22)_{3-6}$; for this reason they can be left out of our field equations.

It remains now to investigate the concavity of the function h' and this will be the argument of the next subsection.

3.1. On the Concavity of h'

Let us see under what conditions the quadratic from $Q = \frac{\partial^2 h'}{\partial\lambda_A \partial\lambda_A}\delta\lambda_A\delta\lambda_B$ is negative defined. Now we have

$$Q = \delta\left(\frac{\partial h'}{\partial\hat{\mu}}\right)\delta(\hat{\mu} + \lambda_{ll} v^2) + \delta\left(\frac{\partial h'}{\partial\hat{\mu}} v^i\right)\delta(-2\lambda_{ll} v^i) + \delta\left(\frac{\partial h'}{\partial\lambda_{ll}} + \frac{\partial h'}{\partial\hat{\mu}} v^2\right)\delta\lambda_{ll}$$

$$+\delta\left(\frac{\partial h'}{\partial\vartheta}\right)\delta\vartheta + \delta\left(\frac{\partial h'}{\partial\mu_i}\right)\delta\mu_i + \delta\left(\frac{\partial h'}{\partial\nu_i}\right)\delta\nu_i$$

After some calculations, it becomes $Q = -2\lambda_{ll} F \delta\upsilon_i \delta\upsilon^i + Q_1$ with

$$Q = \delta\left(\frac{\partial h'}{\partial\hat{\mu}}\right)\delta\hat{\mu} + \delta\left(\frac{\partial h'}{\partial\lambda_{ll}}\right)\delta\lambda_{ll} + \delta\left(\frac{\partial h'}{\partial\vartheta}\right)\partial\vartheta + \delta\left(\frac{\partial h'}{\partial\mu_i}\right)\delta\mu_i + \delta\left(\frac{\partial h'}{\partial\nu_i}\right)\delta\nu_i$$

So Q is negative defined if and only if Q_1 is negative defined. By expliciting its expression, we have

$$Q_1 = 2\frac{\partial h'}{\partial Y_{11}}\delta \mu^i \delta \mu_i + 2\frac{\partial h'}{\partial Y_{22}}\delta v^i \delta v_i + 2\frac{\partial h'}{\partial Y_{12}}\delta \mu^i \delta v_i + Q_2 \text{ with} \tag{24}$$

$$Q_2 - \frac{\partial^2 h'}{\partial \hat{\mu}^2}(\delta\hat{\mu})^2 + 2\frac{\partial^2 h'}{\partial \hat{\mu}\,\partial \lambda_{ll}}\delta \hat{\mu}\,\delta \lambda_{ll} + 2\frac{\partial^2 h'}{\partial \hat{\mu}\,\partial \vartheta}\delta \hat{\mu}\,\delta \vartheta + 4\frac{\partial^2 h'}{\partial \hat{\mu}\,\partial Y_{ll}}\delta \hat{\mu}\,(\mu^i \delta \mu_i) + 2\frac{\partial^2 h'}{\partial \hat{\mu}\,\partial Y_{12}}\delta \hat{\mu}(v^i \delta \mu_i)$$

$$+ 2\frac{\partial^2 h'}{\partial \hat{\mu}\,\partial Y_{12}}\delta \hat{\mu}\,(\mu^i \delta v_i) + 4\frac{\partial^2 h'}{\partial \hat{\mu}\,\partial Y_{22}}\delta \hat{\mu}\,(v^i \delta v_i) + \frac{\partial^2 h'}{\partial \lambda_{ll}^2}(\delta \lambda_{ll})^2 + 2\frac{\partial^2 h'}{\partial \lambda_{ll}\,\partial \vartheta}\delta \lambda_{ll}\partial\vartheta$$

$$+ 4\frac{\partial^2 h'}{\partial \lambda_{ll}\,\partial Y_{11}}\delta \lambda_{ll}\,(\mu^i \delta \mu_i) + 2\frac{\partial^2 h'}{\partial \lambda_{ll}\,\partial \lambda_{12}}\delta \lambda_{ll}\,(v^i \delta \mu_i) + 2\frac{\partial^2 h'}{\partial \lambda_{ll}\,\partial \lambda_{12}}\delta \lambda_{ll}\,(\mu^i \delta v_i)$$

$$+ 4\frac{\partial^2 h'}{\partial \lambda_{ll}\,\partial Y_{22}}\delta \lambda_{ll}\,(v^i \delta v_i) + 2\frac{\partial^2 h'}{\partial \vartheta^2}(\delta \vartheta)2 + 4\frac{\partial^2 h'}{\partial \vartheta\,\partial Y_{11}}\delta \vartheta(\mu^i \delta \mu_i) + 2\frac{\partial^2 h'}{\partial \vartheta\,\partial Y_{12}}\partial \vartheta(v_i \partial \mu_i)$$

$$+ 2\frac{\partial^2 h'}{\partial \vartheta\,\partial Y_{12}}\delta \vartheta(\mu^i \delta v_i) + 4\frac{\partial^2 h'}{\partial \vartheta\,\partial Y_{22}}\delta \vartheta(v^i \partial v_i) + 4\frac{\partial^2 h'}{\partial (Y_{11})^2}(\mu_i \delta \mu^i)^2$$

$$+ 2\frac{\partial^2 h'}{\partial Y_{11}\,\partial Y_{12}}(\mu_i \delta \mu^i)(\mu_j \delta v^j) + 4\frac{\partial^2 h'}{\partial Y_{11}\,\partial Y_{12}}(\mu_i \delta \mu_i)(v_j \delta \mu^j) + 8\frac{\partial^2 h'}{\partial Y_{11}\,\partial Y_{12}}(\mu_i \delta \mu^i)(v_j \delta v^j)$$

$$+ \frac{\partial^2 h'}{\partial (Y_{12})^2}(\mu_i \delta v^i)^2 + 2\frac{\partial^2 h'}{\partial (Y_{12})^2}(\mu_i \delta v^i)(v_j \delta \mu^j) + 4\frac{\partial^2 h'}{\partial Y_{12}\,\partial Y_{22}}(\mu_i \delta v^i)(v_j \delta v^j)$$

$$+ \frac{\partial^2 h'}{\partial (Y_{12})^2}(v_i \delta \mu^i)^2 + 4\frac{\partial^2 h'}{\partial Y_{12}\,\partial Y_{22}}(v_i \delta v^i)(v_j \delta \mu^j) + 4\frac{\partial^2 h'}{\partial (Y_{22})^2}(v_i \delta v^i)^2$$

We evaluate now the coefficients of the differentials in the reference frame where $\vec{\mu}$ and $\vec{v}$ have the components $\vec{\mu} \equiv (\mu_1, 0, 0), \vec{v} \equiv (v_1, v_2, 0)$; the terms containing $\delta\mu_3$ or δv_3 are

$$2\frac{\partial h'}{\partial Y_{11}}(\delta\mu_3)^2 + 2\frac{\partial h'}{\partial Y_{22}}(\delta v_3)^2 + 2\frac{\partial h'}{\partial Y_{12}}(\delta \mu_3)(\delta v_3)$$

So we may deduce that Q_1 can be negative definite only if the following matrix is definite negative

$$\begin{pmatrix} 2\dfrac{\partial h'}{\partial Y_{11}} & \dfrac{\partial h'}{\partial Y_{12}} \\ \dfrac{\partial h'}{\partial Y_{12}} & 2\dfrac{\partial h'}{\partial Y_{22}} \end{pmatrix} \tag{25}$$

Regarding the concavity requirement for all the function h', we can resume what follows:

1. A necessary and sufficient condition for the concavity of all the function h' is that the matrix (25) is negative defined and also the quadratic form Q_1 is definite negative.
2. The matrix (25) must be negative definite; this is a necessary condition in order to have that Q_1 is negative definite.
3. A sufficient condition ensuring that Q is negative definite is that the matrix (25) is negative definite and h' is a non convex function of $\hat{\mu}, \lambda_{ll}, \vartheta, Y_{11}, Y_{12}, Y_{22}$.

In fact, the expression Q_1 is sum of

$$2\frac{\partial h'}{\partial X_{11}}\,\delta\mu_i\,\delta\mu^i + 2\frac{\partial h'}{\partial Y_{22}}\,\delta v_i\,\delta v^i + 2\frac{\partial h'}{\partial Y_{12}}\,\delta\,\mu_i\,\delta v^i \tag{26}$$

(which is negative defined in our hypothesis) and of Q_2. If we evaluate the coefficients of its differentials in the above reference frame where $\vec{\mu}$ and $\vec{v}$ have the components $\vec{\mu} \equiv (\mu_1,\, 0,\, 0)$, $\vec{v} \equiv (v_1, v_2,\, 0)$, we see that its expression is equivalent to the following one

$$Q_2 = \delta\left(\frac{\partial h'}{\partial\hat{\mu}}\right)\delta\,\hat{\mu} + \delta\left(\frac{\partial h'}{\partial\lambda_u}\right)\delta\lambda_u + \delta\left(\frac{\partial h'}{\partial\vartheta}\right)\delta\,\vartheta + \delta\left(\frac{\partial h'}{\partial Y_{11}}\right)\delta\,Y_{11} + \delta\left(\frac{\partial h'}{\partial Y_{12}}\right)\delta\,Y_{12} + \delta\left(\frac{\partial h'}{\partial Y_{22}}\right)\delta\,Y_{22}$$

In our hypothesis their sum is not positive; if it is zero, then both the quadratic forms are zero. The first one of these will imply $\delta\mu_i = 0,\ \delta v^i = 0$. By subtituting this result in the second one, we obtain $\delta\hat{\mu} = 0,\ \delta\lambda_{II} = 0,\ \delta\vartheta = 0$.

It is important to remark this aspect because, in the case of an homogeneous and isotropic media with constant electric permittivity and magnetic permeability, we have that $\delta\left(\frac{\partial h'}{\partial Y_{11}}\right)\delta Y_{11} + \delta\left(\frac{\partial h'}{\partial Y_{12}}\right)\delta Y_{12} + \delta\left(\frac{\partial h'}{\partial Y_{22}}\right)\delta Y_{22}$ is not negative definite but it is identically zero; but our condition is satisfied also in this case because h' is concave function of $\hat{\mu},\ \lambda_{II},\ \vartheta$ and because (26) is negative defined. To explicitate better this particular case, we treat it in the folowing subsection.

3.2. The Particular Case of an Homogeneous and Isotropic Media

In this case we have

$$E^i = \frac{1}{\varepsilon_0}D^i,\ H^i = \frac{1}{\mu_0}B^i \tag{27}$$

By substituting these E^i and H^i in (5) and by using $(17)_{4,5}$, we find

$$-\varepsilon^i = \frac{1}{\varepsilon_0}\frac{\partial h'}{\partial v_i} + \varepsilon^{iab}v_a\frac{\partial h'}{\partial\mu_b},\quad X^i = -\frac{1}{\mu_0}\frac{\partial h'}{\partial\mu_i} + \varepsilon^{iab}v_a\frac{\partial h'}{\partial v_b}$$

By using $(16)_{2,3}$ these expressions become

$$-h_3 v^i = \frac{1}{\varepsilon_0}\frac{\partial h'}{\partial v_i} + \varepsilon^{iab}v_a\frac{\partial h'}{\partial\mu_b},\quad h_3\mu^i = -\frac{1}{\mu_0}\frac{\partial h'}{\partial\mu_i} + \varepsilon^{iab}v_a\frac{\partial h'}{\partial v_b} \tag{28}$$

The dependence of these equations on the velocity v_a shows that Equations (27) hold only in the reference frame comoving with the fluid. So Equations (28) must be replaced with their values in $v_a = 0$, i.e.,

$$-h_3 v^i = \frac{1}{\varepsilon_0}\frac{\partial h'}{\partial v_i},\quad h_3\mu^i = -\frac{1}{\mu_0}\frac{\partial h'}{\partial\mu_i}$$

from which it follows:

$$h' = -\frac{1}{2}h_3(\varepsilon_0 v_i v^i + \mu_0\mu_i\mu^i) + h^*(\hat{\mu},\lambda_{II},\vartheta) = -\frac{1}{2}h_3(\varepsilon_0 Y_{22} + \mu_0\,Y_{11}) + h^*(\hat{\mu},\lambda_{II},\vartheta) \tag{29}$$

After that, Equations $(17)_{4,5}$ give $B^i = -\mu_0 h_3\mu^i,\ D^i = -\varepsilon_0 h_3 v^i$; for Equations $(16)_{2,3}$ these expressions become $B^i = -\mu_0 X^i,\ D^i = -\varepsilon_0\varepsilon^i$, i.e., for Equations (5), $B^i = \mu_0\left[H^i - \left(\vec{v}\wedge\vec{D}\right)^i\right],\ D^i = \varepsilon_0\left[E^i + \left(\vec{v}\wedge\vec{B}\right)^i\right]$.

So we have found

$$\vec{E}=\frac{1}{\varepsilon_0}\vec{D}-\vec{v}\wedge\vec{B},\ \vec{H}=\frac{1}{\mu_0}\vec{B}+\vec{v}\wedge\vec{D} \tag{30}$$

which generalize Equations (27) to any reference frame and coincide with them in the reference frame comoving with the fluid.

By using (29), we see that the matrix (25) becomes

$$\begin{pmatrix} -h_3\,\mu_0 & 0 \\ 0 & -h_3\,\mu_0 \end{pmatrix}$$

which is definite negative if $h_3 > 0$. In the particular case with $h_3 = 1$, the quadratic form Q, considered above, becomes

$$Q=-2\lambda_{II}F\delta v_i\,\delta v^i+\delta\left(\frac{\partial h^*}{\partial\hat{\mu}}\right)\delta\hat{\mu}+\delta\left(\frac{\partial h^*}{\partial\lambda_{II}}\right)\delta\lambda_{II}+\delta\left(\frac{\partial h^*}{\partial\vartheta}\right)\delta\vartheta-h_3\,\mu_0\,\delta\mu_i\delta\mu^i-h_3\varepsilon_0\delta v_i\,\delta v^i$$

which is clearly negative defined.

4. Wave Equations for Maxwell's Equations

let us consider the wave equations for the system (21) in the case

$$h'=h^*\left(\hat{\mu},\lambda_{II},\vartheta\right)+h_1(Y_{11})+h_2(Y_2) \tag{31}$$

Taking into account of (17) and of $(7)_2$, they are

$$(-\lambda+\vec{v}.\vec{n})d\left(\frac{\partial h^*}{\partial\hat{\mu}}\right)+\frac{\partial h^*}{\partial\hat{\mu}}\vec{n}.d\,\vec{v}=0$$

$$(-\lambda+\vec{v}.\vec{n})d\left(\frac{\partial h^*}{\partial\hat{\mu}}v^i\right)+\frac{\partial h^*}{\partial\hat{\mu}}v^i\vec{n}.d\vec{v}-n^i d\left(\frac{h^*+h_1+h_2}{2\lambda_{II}}\right)$$

$$+\frac{\partial h_1}{\partial Y_{11}}(\vec{\mu}.\vec{n})d\left(\frac{\mu^i}{\lambda_{II}}\right)+\frac{\partial h_2{}^*}{\partial Y_{22}}(\vec{v}.\vec{n})d\left(\frac{v^i}{\lambda_{II}}\right)=0$$

$$(-\lambda+\vec{v}.\vec{n})d\left(\frac{\partial h^*}{\partial\hat{\mu}}v^2+\frac{\partial h^*}{\partial\lambda_{II}}\right)+\left(\frac{\partial h^*}{\partial\hat{\mu}}v^2+\frac{\partial h^*}{\partial\lambda_{II}}\right)\vec{n}.\,d\vec{v}-\vec{n}.\,d\left(\frac{h^*+h_1+h_2}{\lambda_{II}}\vec{v}\right)$$

$$+2\,\frac{\partial h_1}{\partial Y_{11}}(\vec{\mu}.\vec{n})d\left(\frac{\mu^r v_r}{\lambda_{II}}\right)+2\left(\frac{\partial h_2}{\partial Y_{22}}\right)(\vec{v}.\vec{n})\,d\left(\frac{v^r v_r}{\lambda_{II}}\right)=0$$

$$(-\lambda+\vec{v}.\vec{n})d\left(\frac{\partial h^*}{\partial\vartheta}\right)+\frac{\partial h^*}{\partial\vartheta}\vec{n}.d\,\vec{v}=0$$

$$(-\lambda+\vec{v}.\vec{n})d\left(2\frac{\partial h_1}{\partial Y_{11}}\mu^i\right)+2\frac{\partial h_1}{\partial Y_{11}}\mu^i\vec{n}.d\,\vec{v}-2\frac{\partial h_1}{\partial Y_{11}}(\vec{\mu}.\vec{n})dv^i+n_k\in^{kis}dv_s=0$$

$$(-\lambda+\vec{v}.\vec{n})d\left(2\frac{\partial h_2}{\partial Y_{22}}v^i\right)+2\frac{\partial h_2}{\partial Y_{22}}v^i\vec{n}.d\,\vec{v}-2\frac{\partial h_2}{\partial Y_{22}}(\vec{v}.\vec{n})dv^i+n_k\in^{ksi}d\mu_s=0 \tag{32}$$

Now we add to $(32)_2$ the equation $(32)_1$ multiplied by $-v^i$; so it becomes

$$(-\lambda+\vec{v}.\vec{n})\frac{\partial h^*}{\partial\hat{\mu}}d(v^i)-n^i d\left(\frac{h^*+h_1+h_2}{2\lambda_{II}}\right)+\frac{\partial h_1}{\partial Y_{11}}(\vec{\mu}.\vec{n})d\left(\frac{\mu^i}{\lambda_{II}}\right)+\frac{\partial h_2}{\partial Y_{22}}(\vec{v}.\vec{n})d\left(\frac{v^i}{\lambda_{II}}\right)=0 \tag{33}$$

Similarly, we add to $(32)_3$ the Equation $(32)_1$ multiplied by $-v^2$ and the Equation (33) multiplied by $-2v^i$; so it becomes

$$(-\lambda+\vec{v}.\vec{n})\left(\frac{\partial h^*}{\partial\lambda_{II}}\right)+\left(\frac{\partial h^*}{\partial\lambda_{II}}-\frac{h^*+h_1+h_2}{\lambda_{II}}\right)\vec{n}.d\vec{v}$$

$$+\frac{\partial h_1}{\partial Y_{11}}(\vec{\mu}.\vec{n})\frac{\mu^r}{\lambda_{II}}d\upsilon^r+2\frac{\partial h_2}{\partial Y_{22}}(\vec{v}.\vec{n})\frac{v^r}{\lambda_{II}}d\upsilon_r=0 \tag{34}$$

We note that in the new system $(32)_{1,4,5,6}$ - (33) , (34), the velocity is present only through its differential dv^i. We note also that λ is there present only through $-\lambda+\vec{v}\,.\,\vec{n}$ so that we obtain wave velocities relative to the normal component of the fluid velocity. Moreover, $(32)_{1,4}$, (33), (34) multiplied by $-2\lambda_{II}$, $(32)_{5,6}$ can be written in the compact form

$$\begin{pmatrix}
(-\lambda+\vec{v}.\vec{n})\frac{\partial^2 h^*}{\partial\hat{\mu}^2} & (-\lambda+\vec{v}.\vec{n})\frac{\partial^2 h^*}{\partial\hat{\mu}\partial\vartheta} & (-\lambda+\vec{v}.\vec{n})\frac{\partial^2 h^*}{\partial\hat{\mu}\partial\lambda_{II}} & b_{1j} & 0 & 0\\
(-\lambda+\vec{v}.\vec{n})\frac{\partial^2 h^*}{\partial\vartheta\,\partial\hat{\mu}} & (-\lambda+\vec{v}.\vec{n})\frac{\partial^2 h^*}{\partial\vartheta^2} & (-\lambda+\vec{v}.\vec{n})\frac{\partial^2 h^*}{\partial\vartheta\partial\lambda_{II}} & b_{2j} & 0 & 0\\
(-\lambda+\vec{v}.\vec{n})\frac{\partial^2 h^*}{\partial\lambda_{II}\,\partial\hat{\mu}} & (-\lambda+\vec{v}.\vec{n})\frac{\partial^2 h^*}{\partial\lambda_{II}\partial\vartheta} & (-\lambda+\vec{v}.\vec{n})\frac{\partial^2 h^*}{\partial\lambda_{II}^2} & b_{3j} & 0 & 0\\
b_{1i} & b_{2i} & b_{3i} & a_{11}^{ij} & a_{12}^{ij} & a_{13}^{ij}\\
0 & 0 & 0 & a_{12}^{ji} & a_{22}^{ji} & a_{23}^{ji}\\
0 & 0 & 0 & a_{13}^{ji} & a_{23}^{ji} & a_{33}^{ji}
\end{pmatrix}
\begin{pmatrix} d\hat{\mu}\\ d\vartheta\\ d\lambda_{II}\\ dv^j\\ d\mu^j\\ dv^j\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\\0\\0\end{pmatrix} \tag{35}$$

with

$$b_{1j}=\frac{\partial h^*}{\partial\hat{\mu}}n_j\,,\; b_{2j}=\frac{\partial h^*}{\partial\vartheta}n_j$$

$$b_{3j}=\left(\frac{\partial h^*}{\partial\lambda_{II}}-\frac{h^*+h_1+h_2}{\lambda_{II}}\right)n_j+2\frac{\partial h_1}{\partial Y_{11}}(\vec{\mu}.\vec{n})\frac{\mu_j}{\lambda_{II}}+2\frac{\partial h_2}{\partial Y_{22}}(\vec{v}.\vec{n})\frac{v_j}{\lambda_{11}}$$

$$a_{11}^{ij}=-2\lambda_{II}(-\lambda+\vec{v}.\vec{n})\frac{\partial h^*}{\partial\vec{\mu}}\delta^{ij}\,,\; a_{12}^{ij}=\frac{\partial h_1}{\partial Y_{11}}n^i\mu_j-2\frac{\partial h_1}{\partial Y_{11}}(\vec{\mu}.\vec{n})\delta^{ij}$$

$$a_{13}^{ij}=2\frac{\partial h_2}{\partial Y_{22}}n^i v_j-2\frac{\partial h_2}{\partial Y_{22}}(\vec{v}.\vec{n})\delta^{ij}\,,\; a_{22}^{ij}=(-\lambda+\vec{v}.\vec{n})\left(2\frac{\partial h_1}{\partial Y_{11}}\delta^{ij}+4\frac{\partial^2 h_1}{\partial Y_{11}^2}\mu^i\mu^j\right)$$

$$a_{23}^{ij}=n_k\in^{kij}\,,\; a_{33}^{ij}=(-\lambda+\vec{v}.\vec{n})\left(2\frac{\partial h_2}{\partial Y_{22}^2}\delta^{ij}+4\frac{\partial^2 h_2}{\partial Y_{22}^2}v^i v^j\right)$$

It is here evident the symmetric form of the system, even if we are not using all the Lagrange multipliers as variables.

We find immediately a first eigenvalue of the problem, i.e., $\boxed{\lambda=\vec{v}.\vec{n}\,.}$

In fact, for every solution of the equation $\frac{\partial h^*}{\partial\hat{\mu}}d\hat{\mu}+\frac{\partial h^*}{\partial\vartheta}d\vartheta=0$ different from zero, we have that $d\hat{\mu}, d\vartheta, d\lambda_{II}=0,\;\; dv^j=0,\; d\mu^j=0,\; dv^j=0$ is an eigenvector of the problem corresponding to this eigenvalue

(We note that, from $(20)_{5,6}$ it follows $dB^i = 2\frac{\partial h_1}{\partial Y_{11}} d\mu^i + 4\frac{\partial^2 h_1}{\partial Y_{11}^2}\mu^i \mu_j d\mu^j$ and $dD^i = 2\frac{\partial h_2}{\partial Y_{22}} dv^i + 4\frac{\partial^2 h_1}{\partial Y_{22}^2} v^i v_j dv^j$ which are zero in the present case; so even if we have not imposed the constraints $n_k dB^k = 0,\ n_k dD^k = 0$ corresponding to Equations $(1)_{7,8}$, they came out from the other equations). This property holds also in absence of the electromagnetic field; it is true that in this case the variable $d\vartheta$ isn't present, but in this case we have also $b_{3j} = \left(\frac{\partial h^*}{\partial \lambda_{11}} - \frac{h^*}{\lambda_{II}}\right) n_j$ so that, for every solution of the system

$$\frac{\partial h^*}{\partial \hat{\mu}} d\hat{\mu} + \left(\frac{\partial h^*}{\partial \lambda_{II}} - \frac{h^*}{\lambda_{II}}\right) d\lambda_{II} = 0,\ n_j dv^j = 0$$

we have a solution of the system; since there are 3 free unknowns, we have that $\lambda = \vec{v}.\vec{n}$ is an eigenvalue with multiplicity at least 3.

If $\lambda \neq \vec{v}.\vec{n}$, the last two equations of the system (35), contracted with n_i, give

$\left(2\frac{\partial h_1}{\partial Y_{11}} n_j \mid + 4\frac{\partial^2 h_1}{\partial Y_{11}^2}(\vec{\mu}.\vec{n})\mu_j\right) d\mu^j = 0$ and $\left(2\frac{\partial h_2}{\partial Y_{22}} n_j + 4\frac{\partial^2 h_2}{\partial Y_{22}^2}(\vec{v}.\vec{n}) v_j\right) dv^j = 0$; so also in the general case, even if we have not imposed the constraints $n_k dB^k = 0,\ n_k dD^k = 0$ corresponding to Equations $(1)_{7,8}$, they came out from the other equations.

To find the eigenvalues with $\lambda \neq \vec{v}.\vec{n}$ we can obtain $d\hat{\mu}, d\vartheta, d\lambda_{II}$ from the first 3 equations of the system (35) and substitute them in the last 3 equations for the determination of $\lambda,\ dv^j, d\mu^j, dv^j$.

To avoid too complicated expressions, we prefer to consider the particular case with $\vec{\mu}.\vec{n} = 0,\ \vec{v}.\vec{n} = 0$ and (29) with $h_3 = 1$ and ε_0, μ_0 constant. In this way we have $h_1 = -\frac{1}{2}\mu_0 Y_{11},\ h_2 = -\frac{1}{2}\varepsilon_0 Y_{22},\ B^i = -\mu_0 \mu^i,\ D^i = -\varepsilon_0 v^i$.

We evaluate our equations in the reference frame with $\vec{n} \equiv (1,0,0)$. Here $\vec{\mu}.\vec{n} = 0,\ \vec{v}.\vec{n} = 0$ become $\mu^1 = 0,\ v^1 = 0$.

Coming back to the previous eigenvalue $\lambda = \vec{v}.\vec{n}$, we see that in the present case the first 3 equations of the system (35) are equivalent to $dv^1 = 0$, Equation $(35)_4$ with $i = 2, 3$ are identities, $(35)_5$ is equivalent to $dv^2 = 0$, $dv^3 = 0$ and $(35)_6$ is equivalent to $d\mu^2 = 0$, $d\mu^3 = 0$, while Equation $(35)_4$ with $i = 1$ becomes

$$\frac{\partial h^*}{\partial \hat{\mu}} d\hat{\mu} + \frac{\partial h^*}{\partial \vartheta} d\vartheta + \left(\frac{\partial h^*}{\partial \lambda_{II}} - \frac{h^* + h_1 + h_2}{\lambda_{II}}\right) d\lambda_{II} = 0$$

So we have only a scalar equation on the 7 unknowns $d\hat{\mu}, d\vartheta, d\lambda_{II}, dv^2, dv^3, d\mu^1, dv^1$. We conclude that the eigenvalue $\lambda = \vec{v}.\vec{n}$ has multiplicity 6 in the present case.

Two other eigenvalues are

$$\boxed{\lambda = \vec{v}.\vec{n} + \frac{1}{\sqrt{\varepsilon_0 \mu_0}},\ \lambda = \vec{v}.\vec{n} - \frac{1}{\sqrt{\varepsilon_0 \mu_0}}} \tag{36}$$

In fact, if $d\mu^2,\ d\mu^3$ are linked only by

$$[\vec{\mu} - \varepsilon_0(-\lambda + \vec{v}.\vec{n})\vec{n} \wedge \vec{v}].d\vec{\mu} = 0$$

we see that $d\vec{\mu}=0,\ d\vartheta=0, d\lambda_{ll}=0,\ dv^j=0,\ d\mu^1=0,\ dv^1=0,\ \ dv^2=-\mu_0(-\lambda+\vec{v}.\vec{n})d\mu^3$, $dv^3=\mu_0(-\lambda+\vec{v}.\vec{n})d\mu^2$ is an eigenvector of the problem. Moreover, $\vec{\mu}+\sqrt{\frac{\varepsilon_0}{\mu_0}}\vec{n}\wedge\vec{v}=\vec{0}$, for the eigenvalue $(36)_1$ there is no constraint on $d\mu^2$ and $d\mu^3$ so that it has at least multiplicity 2.

Similarly, if $\vec{\mu}-\sqrt{\frac{\varepsilon_0}{\mu_0}}\vec{n}\wedge\vec{v}=\vec{0}$, for the eigenvalue $(36)_2$ there is no constraint on $d\mu^2$ and $d\mu^3$ so that it has at least multiplicity 2. The result (36) is very important because, as we have said in the introduction, experiments lead to consider $\frac{1}{\sqrt{\epsilon_0\ \mu_0}}$ as the speed of light in the material; here we have found that it is true except that it is the relative velocity with respect to relative reference frame comoving with the fluid or, more precisely, with respect to the material wave front. This fact shows the reasonableness in choosing $h_3 = 1$, otherwise this result would not have been achieved.

To find other eigenvalues besides $\lambda=\vec{v}.\vec{n}$ and (36), we note that $(35)_4$ with $i = 2, 3$ give $dv^2 = 0$, $dv^3 = 0$ and $(35)_{5,6}$ with $i = 1$ give $d\mu^1 = 0$, $dv^1 = 0$ (as expected). After that, $(35)_{5,6}$ with $i = 2, 3$ become

$$
\begin{aligned}
&-\mu_0\mu^2 dv^1-\mu_0(-\lambda+\vec{v}.\vec{n})d\,\mu^2+d\,v^3=0,\\
&-\mu_0\mu^3 dv^1-\mu_0(-\lambda+\vec{v}.\vec{n})d\,\mu^3-d\,v^2=0,\\
&-\varepsilon_0 v^2 dv^1-\varepsilon_0(-\lambda+\vec{v}.\vec{n})d\,v^2-d\mu^3=0,\\
&-\varepsilon_0 v^3 dv^1-\varepsilon_0(-\lambda+\vec{v}.\vec{n})dv^3+d\mu^3=0
\end{aligned}
\tag{37}
$$

We use the first two of these equations to obtain dv^2, dv^3 and substitute them in the other two equations from which we obtain

$$
\begin{aligned}
d\vec{\mu}&=\left[1-\varepsilon_0\mu_0(-\lambda+\vec{v}.\vec{n})^2\right]^{-1}\left[-\vec{n}\wedge\vec{v}+\mu_0(-\lambda+\vec{v}.\vec{n})\,\vec{\mu}\right]\varepsilon_0 d\,v^1,\\
d\,\vec{v}&=\left[1-\varepsilon_0\mu_0(-\lambda+\vec{v}.\vec{n})^2\right]^{-1}\left[\vec{n}\wedge\vec{\mu}+\varepsilon_0(-\lambda+\vec{v}.\vec{n})\vec{v}\right]\mu_0 d\,v^1,
\end{aligned}
\tag{38}
$$

(The second one of these equations expresses $(37)_{1,2}$ modified by using $(38)_1$; we observe also that (38) multiplied scalarly with $\vec{n}$ shows that d $d\mu^1=0,\ dv^1=0$ are contained in (38)). By substituting these results in $(35)_{1-3}$ and $(35)_4$ with $i = 1$, we obtain the system

$$
\begin{pmatrix}
\frac{\partial^2 h^*}{\partial\hat{\mu}^2} & \frac{\partial^2 h^*}{\partial\hat{\mu}\,\partial\vartheta} & \frac{\partial^2 h^*}{\partial\hat{\mu}\partial\lambda_{ll}} & \frac{\frac{\partial h^*}{\partial\hat{\mu}}}{-\lambda+\vec{v}.\vec{n}}\\
\frac{\partial^2 h^*}{\partial\vartheta\,\partial\hat{\mu}} & \frac{\partial^2 h^*}{\partial\vartheta^2} & \frac{\partial^2 h^*}{\partial\vartheta\,\partial\lambda_{ll}} & \frac{\frac{\partial h^*}{\partial\vartheta}}{-\lambda+\vec{v}.\vec{n}}\\
\frac{\partial^2 h^*}{\partial\lambda_{ll}\,\partial\hat{\mu}} & \frac{\partial^2 h^*}{\partial\lambda_{ll}\,\partial\vartheta} & \frac{\partial^2 h^*}{\partial\lambda_{ll}^2} & \frac{\frac{\partial h^*}{\partial\lambda_{ll}}-\frac{h^*+h_1+h_2}{\lambda_{ll}}}{(-\lambda+\vec{v}.\vec{n})}\\
\frac{\partial^2 h^*}{\partial\hat{\mu}} & \frac{\partial h^*}{\partial\vartheta} & \frac{\partial h^*}{\partial\lambda_{ll}}-\frac{h^*+h_1+h_2}{\lambda_{ll}} & \frac{-2\vec{v}\wedge\vec{\mu}.\vec{n}+(-\lambda+\vec{v}.\vec{n})\overset{-2\lambda_{ll}(-\lambda+\vec{v}.\vec{n})-\mu_0\varepsilon_0}{\left[\mu_0(\vec{\mu})^2+\varepsilon_0(\vec{v})^2\right]}}{1-\varepsilon_0\mu_0(-\lambda+\vec{v}.\vec{n})^2}
\end{pmatrix}
\begin{pmatrix} d\hat{\mu}\\ d\vartheta\\ d\lambda_{ll}\\ dv^1\end{pmatrix}
=
\begin{pmatrix}0\\0\\0\\0\end{pmatrix}
\tag{39}
$$

where we have divided the first 3 equations by $(-\lambda+\vec{v}.\vec{n})$. To express the determinant of the matrix on the left hand side, let us call A_{i4} the algebraic complements of the elements in its fourth coulumn; after that, this determinant becomes $(-\lambda+\vec{v}.\vec{n})^{-1}\left[1-\varepsilon_0\mu_0(-\lambda+\vec{v}.\vec{n})^{-2}\right]^{-1}$ multiplied by the left hand side of the following equation

$$\left[1-\varepsilon_0\mu_0(-\lambda+\vec{v}.\vec{n})^2\right]\left[\frac{\partial h^*}{\partial\hat{\mu}}A_{14}+\frac{\partial h^*}{\partial\vartheta}A_{24}+\left(\frac{\partial h^*}{\partial\lambda_{II}}-\frac{h^*+h_1+h_2}{\lambda_{II}}\right)A_{34}\right]$$

$$-2\lambda_{II}(-\lambda+\vec{v}.\vec{n})^2\left[1-\varepsilon_0\mu_0(-\lambda+\vec{v}.\vec{n})^2\right]A_{44}+2\varepsilon_0\mu_0\vec{v}\wedge\vec{\mu}.\vec{n}(-\lambda+\vec{v}.\vec{n})A_{44}$$

$$-\varepsilon_0\mu_0(-\lambda+\vec{v}.\vec{n})^2\left[\mu_0(\vec{\mu})^2+\varepsilon_0(\vec{v})^2\right]A_{44}=0 \tag{40}$$

which is an equation to determine the last eigenvalues.

We note that, if $\vec{\mu}=a\sqrt{\frac{\varepsilon_0}{\mu_0}}\vec{n}\wedge\vec{v}$ with $a=+1$, the last two terms of (40) become

$2(\varepsilon_0)\mu_0(\vec{v})^2(-\lambda+\vec{v}.\vec{n})A_{44}\left(-\lambda+\vec{v}.\vec{n}+\frac{a}{\sqrt{\varepsilon_0\mu_0}}\right)$ which is zero for $\lambda=\vec{v}.\vec{n}+\frac{a}{\sqrt{\varepsilon_0\mu_0}}$ which is a root also of the other temrs of (40). Vice versa, if we calculate Equation (40) in $\lambda=\vec{v}.\vec{n}+\frac{a}{\sqrt{\varepsilon_0\mu_0}}$, it becomes

$$-A_{44}\left[-2a\sqrt{\varepsilon_0\mu_0}\left(v^2\mu^3-v^3\mu^2\right)+\mu_0\left(\mu^2\right)^2+\mu_0\left(\mu^3\right)^2+\varepsilon_0\left(v^2\right)^2+\varepsilon_0\left(v^3\right)^2\right]$$

$$=-A_{44}\left(\sqrt{\mu_0}\mu^2+a\sqrt{\varepsilon_0}v^3\right)^2+\left(\sqrt{\mu_0}\mu^3-a\sqrt{\varepsilon_0}v^2\right)^2=0$$

Since A_{44} is negative defined, this is possible only if $\mu^2=-a\sqrt{\frac{\varepsilon_0}{\mu_0}}v^3$, $\mu^3=a\sqrt{\frac{\varepsilon_0}{\mu_0}}v^2$, i.e., if $\vec{\mu}=a\sqrt{\frac{\varepsilon_0}{\mu_0}}\vec{n}\wedge\vec{v}$. So this is the only case in which (36) and (40) have a common root.

Finally, we observe now that the other eigenvalue $\lambda=\vec{v}.\vec{n}$ isn't a root of (40). To prove this fact, let us call $\|b_{ij}\|$ the matrix extracted from that on (39) by dropping out its last line and its last coulumn and let us call $|B_{ij}\|$ its adjoint matrix matrix, i.e., with B_{ij} the algebraic complement of b_{ij}; moreover, let be X^i defined by $X^1=\frac{\partial h^*}{\partial\hat{\mu}}$, $X^2=\frac{\partial h^*}{\partial\vartheta}$, $X^3=\frac{\partial h^*}{\partial\lambda_{II}}-\frac{h^*+h_1+h_2}{\lambda_{II}}$. Then Equation (39) calculated in $\lambda=\vec{v}.\vec{n}$ becomes

$$0=\frac{\partial h^*}{\partial\hat{\mu}}A_{14}+\frac{\partial h^*}{\partial\vartheta}A_{24}+\left(\frac{\partial h^*}{\partial\lambda_{II}}-\frac{h^*+h_1+h_2}{\lambda_{II}}\right)A_{34}=\sum_{i,j}^{1,\dots,3}B_{ij}X^iX^j$$

and this is impossibile because $X^1=\frac{\partial h^*}{\partial\hat{\mu}}=F\neq 0$, the matrix $\|b_{ij}\|$ is negative definite and, as a consequence, its adjoint $\|B_{ij}\|$ is positive definite.

So we have found that

- If $\vec{\mu}=\sqrt{\frac{\varepsilon_0}{\mu_0}}\vec{n}\wedge\vec{v}$, then the wave velocities are $\lambda=\vec{v}.\vec{n}$ with multiplicity 6, $\lambda=\vec{v}.\vec{n}-\frac{1}{\sqrt{\varepsilon_0\mu_0}}$ with multiplicity 2, $\lambda=\vec{v}.\vec{n}+\frac{1}{\sqrt{\varepsilon_0\mu_0}}$ with multiplicity 1 and the other 3 eigenvalues of (40),

- If $\vec{\mu} = \sqrt{\frac{\varepsilon_0}{\mu_0}}\vec{n}\wedge\vec{v}$, then the wave velocities are $\lambda = \vec{v}.\vec{n}$ with multiplicity 6, $\lambda = \vec{v}.\vec{n} + \frac{1}{\sqrt{\varepsilon_0\mu_0}}$ with multiplicity 2, $\lambda = \vec{v}.\vec{n} - \frac{1}{\sqrt{\varepsilon_0\mu_0}}$ with multiplicity 1 and the other 3 eigenvalues of (40),

- If $\vec{\mu} \neq \pm\sqrt{\frac{\varepsilon_0}{\mu_0}}\vec{n}\wedge\vec{v}$, then the wave velocities are $\lambda = \vec{v}.\vec{n}$ with multiplicity 6, $\lambda = \vec{v}.\vec{n} \pm \frac{1}{\sqrt{\varepsilon_0\mu_0}}$ with multiplicity 1, and the other 4 eigenvalues of (40).

5. Comparison with Previous Notable Results in Literature

First of all we emphasize that in all the articles cited below the Maxwell Equations are not coupled with the field equations for materials, so that the results obtained in the present article are more general and contain them only as particular cases.

- The paper, Born and Infeld (1934) is very important but seems to belong to the context of general relativity and of quantistic mechanics; moreover, it is connected to the string theory. One of its result is the Born-Infeld Lagrangian and the upper bound $E^2 \leq k$, with k = const. > 0.

  We avoid this framework because the field equations for materials must belong to the same context and literature for them is present in a simpler framework. So the present work is partially less sophisticated than this one for what regards the electromagnetic component of the field equations, even if it leaves out no aspect of Born and Infeld (1934); but our work is anyway more general because it contains also the component of field equations for materials. For this reason it was necessary to reach a compromise between the two and use the same notation for both sets of field equations. We also preferred leaving the notation at a level that would allow practical applications more easily. It is not excluded that this work can be implemented in the future to achieve the same level of refinement.

- In Donato and Ruggeri (1972) discontinuity equations are discussed but with Maxwell Equations without current, free charge and without the field equations for materials. For this ground their results can be compared only with the present ones calculated in d $d\hat{\mu} = 0,\ d\lambda_{II} = 0,\ d\vartheta = 0, d\,\vec{v} = 0,\ \vec{v} = 0$.

  After that we see that their closure is just a special case of the present with $\vec{B} = \mu(H^2)\vec{H}$ and $\vec{D} = \in \vec{E}$ where is considered constant.

  We note the condition (3') which is assumed on page 289 of this article. We see now that it is a consequence of the present concavity requirement. In fact, the present model gives that of Donato and Ruggeri (1972) when h' has the form $h' = -\frac{1}{2}Y_{22} + F(Y_{11})$ and it follows $\mu(H^2) = 2\frac{\partial F}{\partial Y_{11}}$. Consequently, the requirement for the matrix (25) to be positive definite becomes $\in > 0, \mu < 0$. But also (24) must be negative definite; in the present case this condition becomes

$$\left(4\frac{\partial^2 F}{\partial Y_{11}^2}H_iH_j + 2\frac{\partial F}{\partial Y_{11}}\delta_{ij}\right)\delta H_i\delta H_j - \in \delta E_i\delta E_i$$

$$= \left(4\frac{\partial^2 F}{\partial Y_{11}^2}H^2 + 2\frac{\partial F}{\partial Y_{11}}\right)(\delta H_1)^2 + 2\frac{\partial F}{\partial Y_{11}}\left[(\delta H_2)^2 + (\delta H_3)^2\right] - \in \delta E_i\delta E_i$$

  where in the second passages we have used the reference frame where $\vec{H} \equiv (H,0,0)$; we see that it is negative definite if and only if

$$4\frac{\partial^2 F}{\partial Y_{11}^2}H^2 + 2\frac{\partial F}{\partial Y_{11}} < 0 \text{ and } \frac{\partial F}{\partial Y_{11}} < 0$$

and this is condition (3') of Donato and Ruggeri (1972).

Also the speeds of propagation wave (14) and (15) of Donato and Ruggeri (1972) correspond to those found here at the end of the previous section but in the particular case $\varepsilon_0 = 0$.

- In Ruggeri (1973) the sufficient conditions which make all discontinuity-wave propagation-speed real and non vanishing are analyzed. The closure $\vec{B} = \mu(H^2)\vec{H}$ and $\vec{D} = \varepsilon(E^2)\vec{E}$ is more general than that of the previous article but less general than the present one. The Pignedoli's conditions 4a) and 4d) on page 285 are compatible with the present concavity of h'; 4b) and 4c) are necessary and sufficient conditions for the existence of h' such that $D^i = \frac{\partial h'}{\partial E_i}$, $B^i = \frac{\partial h'}{\partial H_i}$ which has been found later in Boillat *et al.* (1994), we have proved here that this property holds also in the presence of the field equations for the material. The only difference now is that h' may depend also on mass density and energy density of the material, besides its dependence on the electromagnetic tensor; moreover, the derivatives of h' with respect to E_i and H_i must be replaced by derivatives of h' with respect to ε_i and X_i.

  In Ruggeri (1973), Equations (23) and (31), the wave velocities have been found under the particular assumption that $\vec{n} \wedge \vec{E}$ is orthogonal to $\vec{n} \wedge \vec{H}$. If we write the above wave equations in this hypothesis, we see that it yields the same result.

- In Boillat *et al.* (1994) it is shown how to obtain hyperbolic systems compatible with an entropy, especially when it consists of one scalar and one vectorial function. The Maxwell Equations are considered but without charge-current densities. The generating vector is $h'^i = c\varepsilon^{ijk}E_j H_k$ and is said that one can take $c = 1$, without restriction. By comparing them with (18) we see the same result but calculated in $\vec{v}=0$. (The constant c of this article corresponds to the constant h_3 of the present one). In section E) they use the constitutive relations $\vec{B} = \mu(H^2)\vec{H}$ and $\vec{D} = \in(E^2)\vec{E}$ which are an improvement of that in Donato and Ruggeri (1972) but still a particular case of the present one.

  The article Gibbons and Herdeiro (2001) has the same characteristics of Born and Infeld (1934). from which it starts; for this reason our comments are the same. But in any case we appreciate that the Boillat metric and the spacetime metric are used. In particular, the propagation of fluctuations in a non trivial background field is described by means of two cones, one for the Einstein Geometry and the other one for the effective geometry governed by the Boillat metric.

  Exact stationary solutions are analyzed. Blons are considered, which are static finite energy solutions and differ from solitons for the fact that have distributional sources and also singularities.

  In Gibbons and Herdeiro (2001) Maxwell equations are used only as an application example of several inequalities that have been obtained for the components of

  $$T^\alpha{}_\beta = q^r{}_\beta \frac{\partial \mathcal{L}}{\partial q^r_\alpha} - \delta^\alpha{}_\beta \mathcal{L}, \text{ with } \mathcal{L} \text{ a general Lagrangian function.}$$

  Moreover, wave velocities and characteristic shocks are studied with particular attention to the generalized Born-Infeld Lagrangian. From the last two lines of page 3471 we note that $\vec{E}$ and $\vec{B}$ are taken as Lagrange multipliers, while we have taken $\vec{\in}$ and $\vec{X}$ in this role because this choice preserves the material frame objectivity; obviously, the two choices are possible if calculated in $\vec{v} = \vec{0}$.

Conclusion

Maxwell's equations in materials coupled with Euler equations for polyatomic gases have been here considered. By imposing the existence of a supplementary conservation law, a scalar function h' has been found such that $B^i = \frac{\partial h'}{\partial X_i}$, $D^i = \frac{\partial h'}{\partial \in_i}$ where $-\vec{\in} = \vec{E} + \vec{v} \wedge \vec{B}$ is the electromotive force and $\vec{X} = -\vec{H} + \vec{v} \wedge \vec{D}$ its dual. In this way all the set of field equations can be written in the symmetric hyperbolic form, by using the already known literature on hyperbolic systems with evolutive constraints. Aim of a future research is to find the relativistic counterpart of the present study.

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