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Some Theorems in Existence, Uniqueness and Stability Solutions of Volterra Integro-Differential Equations of the First Order

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Abstract

We study the existence, uniqueness and stability solutions of new Volterra integro-differential equations of the first order by using Picard approximation method, Banach fixed point theorem. Theorems on the existence, uniqueness and stability solutions are established under some necessary and sufficient conditions on closed and bounded domains. Furthermore the study of such nonlinear of Volterra integro-differential equations leads us to improve and extend the above methods and thus the non-linear Volterra integro-differential equations that we have introduced in this study become more general and detailed than those introduced some results by Butris and Rafeq (2011).

Keywords: *Existence, uniqueness and stability solution, Integro-differential equation, Picard approximation method, Banach fixed point theorem*

Introduction

Integro-differential equations of various type and kinds play an important role in many branches of mathematics and engineering field. Analytical solution of this kind of equation is not accessible in general form of equation and we can get an exact solution only in special cases. But in industrial problems we have not spatial cases so that we try to solve this kind of equations numerically in general format. Many numerical schemes are employed to give an approximate solution with sufficient accuracy (Tricomi, 1965; Rama, 1981; Burton, 2005; Palais., 2007; and Mahdi Monje and Ahmed, 2019). Many authors create and develop successive approximation method and Banach fixed point theorem (Struble,1962; Andrzej and James, 2003; Beeker and Burton,2006; Battelli and Feckan., 2008; Butris and Rafeq, 2011; Abdullah, 2015; Manouchehr *et al.*, 2018; and Pakhshan *et al.*, 2019) and schemes to investigate the solution of integral equations describing many applications in mathematical and engineering field.

Definition 1: A function f is defined on a set $E \subseteq R$ is said to be continuous at point x in E if $\epsilon > 0$ is given, there is a a positive number δ, such that for all y in E with $|x - y| < \delta$ we have $|f(x) - f(y)| < \epsilon$.

Definition 2: A continuous function f satisfy a Lipchitz condition on the domain $G = \{(t, x): a \le t \le b, c \le x \le d\}$ in the variable x on G if for all $K > 0$ and $(t, x_1), (t, x_2) \in G$, such that:

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$|f(t,x_1)-f(t,x_2)| \le K|x_1-x_2|$

Definition 3: Let $\{f_i(t)\}_{i=0}^{\infty}$ be a sequence of functions defined on a set $E \subseteq R$. We say that $\{f_i(t)\}_{i=0}^{\infty}$ converges uniformly to the limit function *f* on *E* if $\epsilon > 0$ is given, there exist a positive integer *N* such that:

$|f_i(t)-f(t)| < \epsilon$ for all $i \ge N, t \in E$.

Definition 4: A solution $x(t)$ is said to be stable if for each $\epsilon > 0$ there exist a $\delta > 0$ such that any solution $\bar{x}(t)$ which satisfies $\|\bar{x}(t_0)-x(t_0)\| < \delta$ for some t_0 also satisfies $\|\bar{x}(t)-x(t)\| < \epsilon$ for all $t \ge t_0$.

Definition 5: Let *E* be a vector space of a real-valued function $\|.\|$ of E in to *R* called a norm if satisfies:

1. $\|x\| \ge 0$ for all $x \in E$
2. $\|x\| = 0$ if and only if $x = 0$
3. $\|x+y\| \le \|x\| + \|y\|$ for all $x, y \in E$
4. $\|\alpha x\| = |\alpha|\|x\|$ for all $x \in E$ and $\alpha \in R$

Definition 6: A linear space *E* with a norm defined on it is called a normed space.

Definition 7: A normed linear space *E* is called complete if for every Cauchy sequence in *E* convergent to an element in *E*.

Definition 8: A complete normed linear space is a Banach space.

Definition 9: If T^* map *E* into itself and *v* is a point of *E* such that $T^*_v = v$, then *v* is a fixed point of T^*.

Definition 10: Let $(E, \|.\|)$ be a norm space. If T^* map into itself we say that T^* is a contraction mapping on *E* if there exists $\alpha \in R$ with $0 < \alpha < 1$ such that:

$\|T^*x - T^*y\| \le \alpha\|x-y\|$, $x, y \in E$

Theorem 1 (Banach Fixed Point Theorem): Let $(E, \|.\|)$ be complete metric space with a contraction mapping $T^*: E \to E$. Then T^* a unique fixed-point x in E (i.e. $T,^* x = x$). (For the definitions and theorem see (Butris and Hasso, 2000; Butris and Aziz, 2006; and Butris and Rafeq, 2011).

In this paper, we prove the existence, uniqueness and stability solution of Volterra integro-differential equations of the first order by using both method of Picard approximation and Banach fixed point theorem which are given in (Struble, 1962; and Rama, 1981).

Consider the following integro-differential equations of the first order:

$$\left.\begin{aligned} \frac{dx}{dt} &= Ax + f\left(t,x,y,\int_{-\infty}^{t} R(t-s)x(s)ds\right) \\ \frac{dy}{dt} &= By + g\left(t,x,y,\int_{-\infty}^{t} H(t-s)y(s)ds\right) \end{aligned}\right\} \quad \text{...(1)}$$

Let the vector functions $f(t, x, y, z)$ and $g(t, x, y, v)$ be defined and continuous on the domains:

$$\left.\begin{aligned} (t,x,y,z) &\in R^1 \times D \times D_1 \times D_z \\ (t,x,y,v) &\in R^1 \times D \times D_1 \times D_v \end{aligned}\right\} \quad \text{...(2)}$$

where D, D_1 are closed and bounded domains subsets of Euclidean space R^n and D_z, D_v are bounded domains subset of the Euclidean space R^m.

Suppose that the vector functionsand $f(t, x, y, z)$ and $g(t, x, y, v)$ satisfy the following inequalities:

$$\left.\begin{array}{l}\|f(t,x,y,z)\| \le M\\ \|g(t,x,y,v)\| \le N\end{array}\right\} \qquad ...(3)$$

$$\|f(t,x_1,y_1,z_1) - f(t,x_2,y_2,z_2)\| \le K_1\|x_1 - x_2\| + K_2\|y_1 - y_2\| + K_3\|z_1 - z_2\| \qquad ...(4)$$

$$\|g(t,x_1,y_1,v_1) - g(t,x_2,y_2,v_2)\| \le L_1\|x_1 - x_2\| + L_2\|y_1 - y_2\| + L_3\|v_1 - v_2\| \qquad ...(5)$$

for all $t \in R^1, x, x_1, x_2 \in D, y, y_1, y_2 \in D_1, z, z_1, z_2 \in D_z$ and $v, v_1, v_2 \in D_v$.

where M, N, K_1, K_2, K_3, L_1, L_2, and L_3 are positive constants.

Suppose that $A = [A_{ij}]$ and $B = [B_{ij}]$ are (x n) positive matrices which are continuous in t, and satisfy the following inequalities:

$$\left\|e^{A(t-s)}\right\| \le \delta e^{-\lambda(t-s)} \qquad ...(6)$$

$$\left\|e^{B(t-s)}\right\| \le \varepsilon e^{-\chi(t-s)} \qquad ...(7)$$

$$\|R(t-s)\| \le c\frac{e^{-\lambda(t-s)}}{(t-s)^{1-\alpha}} \qquad ...(8)$$

$$\|H(t-s)\| \le d\frac{e^{-\chi(t-s)}}{(t-s)^{1-\beta}} \qquad ...(9)$$

where $0 < \alpha, \beta \le 1$ and $\delta, \lambda, \varepsilon, \chi, c, d$ are positive constants.

We define the non-empty sets as follows:

$$\left.\begin{array}{l}D_f = D - \dfrac{\delta}{\lambda}(1 - e^{-\lambda T})M\\ D_g = D_1 - \dfrac{\varepsilon}{\chi}(1 - e^{-\chi T})N\end{array}\right\} \qquad ...(10)$$

Furthermore, we suppose that the largest eigen-value of the matrix

$\Lambda = \begin{pmatrix} E_1 & E_2 \\ E_3 & E_4 \end{pmatrix}$ does not exceed unity

$$\lambda_{max}(\Lambda) = \frac{(E_1+E_4) + \sqrt{(E_1+E_4)^2 - 4(E_1 E_4 - E_2E_3)}}{2} < 1 \qquad ...(11)$$

where

$$E_1 = \frac{\delta}{\lambda}(1 - e^{-\lambda T})(K_1 + K_3\frac{cT^{\alpha-1}}{2\lambda}), E_2 = \frac{\delta}{\lambda}(1 - e^{-\lambda T})K_2$$

$$E_3 = \frac{\varepsilon}{\chi}(1 - e^{-\chi T})L_1, \quad E_4 = \frac{\varepsilon}{\chi}(1 - e^{-\chi T})(L_2 + L_3\frac{dT^{\beta-1}}{2\chi})$$

We define the sequence of functions $\{x_m(t), y_m(t)\}_{m=0}^{\infty}$ on the domains (2) by the following:

$$x_{m+1}(t) = x_0 + \int_0^t e^{A(t-s)}[f(s, x_m(s), y_m(s), \int_{-\infty}^{s} R(s-\tau)x_m(\tau)d\tau]ds \qquad ...(12)$$

$x_0(0) = x_0$, for all $m = 0, 1, 2, ...$

and

$$y_{m+1}(t) = y_0 + \int_0^t e^{B(t-s)}[\, g(s, x_m(s), y_m(s), \int_{-\infty}^{s} H(s-\tau)y_m(\tau)d\tau]ds \quad ...(13)$$

$y_0(0) = y_0$, for all $m = 0, 1, 2, ...$

2. Existence Solution of (1)

In this section, we prove the existence theorem of integro-differential equation (1) by using Picard approximation method.

Theorem 2 (Existence Theorem): Let the vector functions $f(t, x, y, z)$ and $g(t, x, y, v)$ are defined and continuous on the domain (2) satisfy the inequalities from (3) to (9) and condition (10). Then there exist the sequence of functions (12) and (13) convergent uniformly on the domain:

$$\left.\begin{matrix}(t, x_0) \in R^1 \times D_f \\ (t, y_0) \in R^1 \times D_g\end{matrix}\right\} \quad ...(14)$$

to the limit functions $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ which satisfy the following integral equations:

$$x(t) = x_0 + \int_0^t e^{A(t-s)}[\, f(s, x(s), y(s), \int_{-\infty}^{s} R(s-\tau)x(\tau)d\tau]ds \quad ...(15)$$

and

$$y(t) = y_0 + \int_0^t e^{B(t-s)}[\, g(s, x(s), y(s), \int_{-\infty}^{s} H(s-\tau)y(\tau)d\tau]ds \quad ...(16)$$

And it's a unique solution of (1), provided that

$$\begin{pmatrix} \|x_m(t) - x_0\| \\ \|y_m(t) - y_0\| \end{pmatrix} \le \begin{pmatrix} \frac{\delta}{\lambda}(1 - e^{-\lambda T})M \\ \frac{\varepsilon}{\overline{\lambda}}(1 - e^{-\overline{\lambda} T})N \end{pmatrix} \quad ...(17)$$

and

$$\begin{pmatrix} \|x(t) - x_m(t)\| \\ \|y(t) - y_m(t)\| \end{pmatrix} \le \Lambda^m (E - \Lambda)^{-1}\Omega_1 \quad ...(18)$$

Proof: Put $m = 0$ in (12), we get

$$\|x_1(t) - x_0\| \le \int_0^t \|e^{A(t-s)}\| \|\, f(s, x_0, y_0, \int_{-\infty}^{s} R(s-\tau)x_0 d\tau]\|ds$$

$\le \frac{\delta}{\lambda}(1 - e^{-\lambda t})M$, for all $t \in [0, T]$

And hence

$$\|x_1(t) - x_0\| \le \frac{\delta}{\lambda}(1 - e^{-\lambda T})M \quad ...(19)$$

Therefore, $x_1(t) \in D$ for all $t \in [0, T]$ and $x_0 \in D_f$.

Also, from (13) when $m = 0$, we have

$$\|y_1(t) - y_0\| \le \int_0^t \|e^{B(t-s)}\| \| g(s, x_0, y_0, \int_{-\infty}^{s} H(s-\tau) y_0 d\tau] \| ds$$

$$\le \frac{\varepsilon}{\Game}(1 - e^{-\Game t})N, \text{ for all } t \in [0, T]$$

Therefore,

$$\|y_1(t) - y_0\| \le \frac{\varepsilon}{\Game}(1 - e^{-\Game T})N \qquad ...(20)$$

Therefore $y_1(t) \in D_1$ for all $t \in R^1, y_0 \in D_g$

Then by mathematical induction, we can prove that:

$$\left.\begin{aligned} \|x_m(t) - x_0\| &\le \frac{\delta}{\lambda}(1 - e^{-\lambda T})M \\ \|y_m(t) - y_0\| &\le \frac{\varepsilon}{\Game}(1 - e^{-\Game T})N \end{aligned}\right\} \qquad ...(21)$$

Therefore $x_m(t) \in D, y_m(t) \in D_1, t \in [0, T], x_0 \in D_f, y_0 \in D_g, m = 1,2, \dots$

By rewriting the inequality (21) by the vector form, then we have (17).

Next, we shall prove that the sequence of functions (12) and (13) convergent uniformly on the domain (2).

When $m = 1$ in (12) and (13), we have

$$\|x_2(t) - x_1(t)\|$$
$$\le \| \int_0^t e^{A(t-s)}[f(s, x_1(s), y_1(s), \int_{-\infty}^{s} R(s-\tau) x_1(\tau) d\tau] ds$$
$$- \int_0^t e^{A(t-s)}[f(s, x_0, y_0, \int_{-\infty}^{s} R(s-\tau) x_0 d\tau] ds \|$$

$$\le \int_0^t \left\|e^{A(t-s)}\right\| [K_1 \|x_1(s) - x_0\| + K_2 \|y_1(s) - y_0\|$$
$$+ K_3 \int_{-\infty}^{s} \|R(s-\tau)\| \|x_1(\tau) - x_0\| d\tau] ds$$

$$\le \int_0^t \left\|e^{A(t-s)}\right\| [K_1 \|x_1(s) - x_0\| + K_2 \|y_1(s) - y_0\| + K_3 \frac{ct^{\alpha-1}}{2\lambda} \|x_1(s)$$
$$- x_0\| ds$$

$$\le \frac{\delta}{\lambda}(1 - e^{-\lambda t})[(K_1 + K_3 \frac{ct^{\alpha-1}}{2\lambda}) \|x_1(t) - x_0\| + K_2 \|y_1(t) - y_0\|]$$

Thus,

$$\|x_2(t) - x_1(t)\| \le E_1(t) \|x_1(t) - x_0\| + E_2(t) \|y_1(t) - y_0\|$$

By the same way above, we get

$$\|y_2(t)-y_1(t)\| \le \int_0^t \|e^{B(t-s)}\| [L_1\|x_1(s)-x_0\| + L_2\|y_1(s)-y_0\| + L_3\int_{-\infty}^{s}\|H(s-\tau)\|\|y_1(\tau)-y_0\|d\tau]ds$$

$$\le \int_0^t \|e^{B(t-s)}\| [L_1\|x_1(s)-x_0\| + L_2\|y_1(s)-y_0\| + L_3\frac{d\,t^{\beta-1}}{2ϫ}\|y_1(s)-y_0\|ds$$

$$\le \frac{\varepsilon}{ϫ}(1-e^{-ϫt})[L_1\|x_1(t)-x_0\| + (L_2+L_3\frac{d\,t^{\beta-1}}{2ϫ})\|y_1(t)-y_0\|]$$

Therefore,

$$\|y_2(t)-y_1(t)\| \le E_3(t)\|x_1(t)-x_0\| + E_4(t)\|y_1(t)-y_0\|$$

Then by the mathematical induction, we can obtain that

$$\|x_{m+1}(t)-x_m(t)\| \le E_1(t)\|x_m(t)-x_{m-1}(t)\| + E_2(t)\|y_m(t)-y_{m-1}(t)\| \quad ...(22)$$

Similarly,

$$\|y_{m+1}(t)-y_m(t)\| \le E_3(t)\|x_m(t)-x_{m-1}(t)\| + E_4(t)\|y_m(t)-y_{m-1}(t)\| \quad ...(23)$$

Rewriting inequalities (22) and (23) by vector form, we have

$$\Omega_{m+1}(t) \le \Lambda(t)\Omega_m \quad ...(24)$$

$$\Omega_{m+1} = \begin{pmatrix} \|x_{m+1}(t)-x_m(t)\| \\ \|y_{m+1}(t)-y_m(t)\| \end{pmatrix}$$

$$\Omega_m = \begin{pmatrix} \|x_m(t)-x_{m-1}(t)\| \\ \|y_m(t)-y_{m-1}(t)\| \end{pmatrix}$$

and

$$\Lambda(t) = \begin{pmatrix} E_1(t) & E_2(t) \\ E_3(t) & E_4(t) \end{pmatrix}$$

Now, we take the maximal value for the both sides of the inequality (24) we have

$$\Omega_{m+1} \le \Lambda\Omega_m \quad ...(25)$$

where $\Lambda = \max_{t\in[0,T]} \Lambda(t)$, we obtain

$$\Lambda = \begin{pmatrix} E_1 & E_2 \\ E_3 & E_4 \end{pmatrix}$$

By repetition (25), we find that

$$\Omega_{m+1} \le \Lambda^m \Omega_1 \quad ...(26)$$

where $\Omega_1 = \begin{pmatrix} \frac{\delta}{\lambda}(1-e^{-\lambda T})M \\ \frac{\varepsilon}{ϫ}(1-e^{-ϫT})N \end{pmatrix}$

Thus,

$$\sum_{i=1}^{m} \Omega_i \leq \sum_{i=1}^{m} \Lambda^{i-1}\Omega_1 \qquad ...(27)$$

By using (11), then the sequence (27) is uniformly convergent that is

$$\lim_{m\to\infty} \sum_{i=1}^{m} \Lambda^{i-1}\Omega_1 = \sum_{i=1}^{\infty} \Lambda^{i-1}\Omega_1 = (E-\Lambda)^{-1}\Omega_1 \qquad ...(28)$$

Let

$$\lim_{m\to\infty} \begin{pmatrix} x_m(t) \\ y_m(t) \end{pmatrix} = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \qquad ...(29)$$

Since the sequence of functions (12) and (13) are define and continuous in the domain (2) then the limiting vector function

$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ is also defined and continuous on the same domain, hence the vector function $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ is a solution of (1)

Theorem 3 (Uniqueness Theorem): With the hypotheses and all conditions and inequalities of the theorem 2, then the solutions $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ is a unique on the domain (2).

Proof: Let $\begin{pmatrix} x^*(t) \\ y^*(t) \end{pmatrix}$ be another solution of (1).

where

$$x^*(t) = x_0 + \int_0^t e^{A(t-s)}[f(s, x^*(s), y^*(s), \int_{-\infty}^{s} R(s-\tau)x^*(\tau)\, d\tau)ds] \qquad ...(30)$$

and

$$y^*(t) = y_0 + \int_0^t e^{B(t-s)}[g(s, x^*(s), y^*(s), \int_{-\infty}^{s} H(s-\tau)y^*(\tau)\, d\tau)ds] \qquad ...(31)$$

Now,

$$\|x(t) - x^*(t)\| \leq \int_0^t \|e^{A(t-s)}\| \left[K_1\|x(s) - x^*(s)\| + K_2\|y(s) - y^*(s)\| + K_3 \frac{ct^{\alpha-1}}{2\lambda}\|x(s) - x^*(s)\|\right] ds$$

Therefore,

$$\|x(t) - x^*(t)\| \leq \frac{\delta}{\lambda}(1 - e^{-\lambda T})\left[(K_1 + K_3\frac{cT^{\alpha-1}}{2\lambda})\|x(t) - x^*(t)\| + K_2\|y(t) - y^*(t)\|\right]$$

Hence,

$$\|x(t)-x^*(t)\| \le E_1\|x(t)-x^*(t)\| + E_2\|y(t)-y^*(t)\| \quad ...(32)$$

And also,

$$\|y(t)-y^*(t)\| \le \int_0^t \|e^{B(t-s)}\| \left[L_1\|x(s)-x^*(s)\| + L_2\|y(s)-y^*(s)\| + L_3\frac{d\,t^{\beta-1}}{2Ӽ}\|y(s)-y^*(s)\|\right] ds$$

Therefore

$$\|y(t)-y^*(t)\| \le \frac{\varepsilon}{Ӽ}(1 - e^{-ӼT})\left[L_1\|x(t)-x^*(t)\| + (L_2 + L_3\frac{dT^{\beta-1}}{2Ӽ})\|y(t)-y^*(t)\|\right]$$

So,

$$\|y(t)-y^*(t)\| \le E_3\|x(t)-x^*(t)\| + E_4\|y(t)-y^*(t)\| \quad ...(33)$$

Rewrite the inequalities (32) and (33) by the vector form:

$$\begin{pmatrix}\|x(t)-x^*(t)\| \\ \|y(t)-y^*(t)\|\end{pmatrix} \le \Lambda \begin{pmatrix}\|x(t)-x^*(t)\| \\ \|y(t)-y^*(t)\|\end{pmatrix} \quad ...(34)$$

Then by the condition (11) we get

$\begin{pmatrix}\|x(t)-x^*(t)\| \\ \|y(t)-y^*(t)\|\end{pmatrix} < \begin{pmatrix}\|x(t)-x^*(t)\| \\ \|y(t)-y^*(t)\|\end{pmatrix}$ we get contradiction,

then

$$\begin{pmatrix}\|x(t)-x^*(t)\| \\ \|y(t)-y^*(t)\|\end{pmatrix} \to \begin{pmatrix}0 \\ 0\end{pmatrix}$$

Therefore,

$$\begin{pmatrix}x(t) \\ y(t)\end{pmatrix} = \begin{pmatrix}x^*(t) \\ y^*(t)\end{pmatrix}$$

and hence the solutions $\begin{pmatrix}x(t) \\ y(t)\end{pmatrix}$ is a unique solution of (1).

Finally, we shall prove that the solution $\begin{pmatrix}x(t) \\ y(t)\end{pmatrix} \in \begin{pmatrix}D \\ D_1\end{pmatrix}$ on the domain (2).

Assuming

$$\Delta_1(t) = \left\|\int_0^t e^{A(t-s)}[f(s,x_m(s),y_m(s),\int_{-\infty}^s R(s-\tau)x_m(\tau)d\tau]ds - \int_0^t e^{A(t-s)}[f(s,x(s),y(s),\int_{-\infty}^s R(s-\tau)x(\tau)d\tau]ds\right\| \le E_1\|x_m(t)-x(t)\| + E_2\|y_m(t)-y(t)\| \quad ...(35)$$

and

$$\Delta_2(t) = \left\| \int_0^t e^{B(t-s)}[f(s, x_m(s), y_m(s), \int_{-\infty}^s H(s-\tau)x_m(\tau)d\tau]ds - \int_0^t e^{B(t-s)}[f(s, x(s), y(s), \int_{-\infty}^s H(s-\tau)x(\tau)d\tau]ds \right\| \le E_3 \|x_m(t) - x(t)\| + E_4 \|y_m(t) - y(t)\| \quad ...(36)$$

Rewrite the inequalities (35) and (36) in a vector form, we have

$$\begin{pmatrix} \Delta_1(t) \\ \Delta_2(t) \end{pmatrix} \le \Lambda \begin{pmatrix} \|x_m(t) - x(t)\| \\ \|y_m(t) - y(t)\| \end{pmatrix}$$

But the sequence of functions (12) and (13) convergent uniformly on the domain (2), therefore $\|x_m(t) - x(t)\| \le \epsilon_1$ and $\|y_m(t) - y(t)\| \le \epsilon_2$

$\epsilon_1, \epsilon_2 \ge 0.$

That is

$$\begin{pmatrix} \Delta_1(t) \\ \Delta_2(t) \end{pmatrix} \le \Lambda \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix}$$

By condition (11), we get the solution $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \in \begin{pmatrix} D \\ D_1 \end{pmatrix}$ on the domain (2)

3. Stability Solution of (1)

In this section, we study the stability solution of the problem (1) by the following theorem:

Theorem 4 (Stability Theorem): If the inequalities (3) to (9) are satisfied and $\begin{pmatrix} \bar{x}(t) \\ \bar{y}(t) \end{pmatrix}$ which are another solutions of (1) then the solutions is stable for all $t \geq 0$.

where

$$\bar{x}(t) = \bar{x}_0 + \int_0^t e^{A(t-s)}[f(s, \bar{x}(s), \bar{y}(s), \int_{-\infty}^s R(s-\tau)\bar{x}(\tau)\,d\tau)ds] \quad ...(37)$$

and

$$\bar{y}(t) = \bar{y}_0 + \int_0^t e^{B(t-s)}[g(s, \bar{x}(s), \bar{y}(s), \int_{-\infty}^s H(s-\tau)\bar{y}(\tau)\,d\tau)ds] \quad ...(38)$$

Proof: Taking

$$\|x(t) - \bar{x}(t)\| = \left\| x_0 e^{At} + \int_0^t e^{A(t-s)}[f(s, x(s), y(s), \int_{-\infty}^s R(s-\tau)x(\tau)\,d\tau)ds] - \bar{x}_0 e^{At} - \int_0^t e^{A(t-s)}[f(s, \bar{x}(s), \bar{y}(s), \int_{-\infty}^s R(s-\tau)\bar{x}(\tau)\,d\tau)ds] \right\|$$

$$\le \|x_0 - \bar{x}_0\| + \int_0^t \|e^{A(t-s)}\| \, [K_1\|x(s) - \bar{x}(s)\| + K_2\|y(s) - \bar{y}(s)\| + K_3 \frac{cT^{\alpha-1}}{2\lambda} \|x(s) - \bar{x}(s)\|]ds$$

Therefore

$$\|x(t)-\bar{x}(t)\|$$
$$\le \|x_0-\bar{x}_0\| + \int_0^t \left\|e^{A(t-s)}\right\|[(K_1 + K_3\frac{cT^{\alpha-1}}{2\lambda})\|x(s)-\bar{x}(s)\|$$
$$+ K_2\|y(s)-\bar{y}(s)\|]ds$$

So

$$\|x(t)-\bar{x}(t)\| \le \|x_0-\bar{x}_0\| + E_1\|x(t)-\bar{x}(t)\| + E_2\|y(t)-\bar{y}(t)\|$$

And according to the definition of stability for $\|x_0-\bar{x}_0\| \le \delta_1$ we get:

$$\|x(t)-\bar{x}(t)\| \le \delta_1 + E_1\|x(t)-\bar{x}(t)\| + E_2\|y(t)-\bar{y}(t)\| \quad ...(39)$$

Also

$$\|y(t)-\bar{y}(t)\|$$
$$= \left\| y_0 + \int_0^t e^{B(t-s)}[g(s,x(s),y(s),\int_{-\infty}^s H(s-\tau)y(\tau)\,d\tau)ds] \right.$$
$$\left. - \bar{y}_0 - \int_0^t e^{B(t-s)}[g(s,\bar{x}(s),\bar{y}(s),\int_{-\infty}^s H(s-\tau)\bar{y}(\tau)\,d\tau)ds] \right\|$$

$$\le \|y_0-\bar{y}_0\| + \int_0^t \left\|e^{B(t-s)}\right\| [L_1\|x(s)-\bar{x}(s)\| + L_2\|y(s)-\bar{y}(s)\|$$
$$+ L_3\frac{dT^{\beta-1}}{2\lambda}\|y(s)-\bar{y}(s)\|]ds$$

Hence

$$\|y(t)-\bar{y}(t)\|$$
$$\le \|y_0-\bar{y}_0\| + \int_0^t \left\|e^{B(t-s)}\right\|[L_1\|x(t)-\bar{x}(t)\| + (L_2$$
$$+ L_3\frac{dT^{\beta-1}}{2\lambda})\|y(t)-\bar{y}(t)\|]ds$$

Thus

$$\|y(t)-\bar{y}(t)\| \le \|y_0-\bar{y}_0\| + E_3\|x(t)-\bar{x}(t)\| + E_4\|y(t)-\bar{y}(t)\|$$

Also by the definition of stability for $\|y_0-\bar{y}_0\| \le \delta_2$ we get:

$$\|y(t)-\bar{y}(t)\| \le \delta_2 + E_3\|x(t)-\bar{x}(t)\| + E_4\|y(t)-\bar{y}(t)\| \quad ...(40)$$

From the inequalities (37) and (38) we have:

$$\begin{pmatrix}\|x(t)-\bar{x}(t)\| \\ \|y(t)-\bar{y}(t)\|\end{pmatrix} \le \begin{pmatrix}\delta_1 \\ \delta_2\end{pmatrix} + \begin{pmatrix}E_1 & E_2 \\ E_3 & E_4\end{pmatrix}\begin{pmatrix}\|x(t)-\bar{x}(t)\| \\ \|y(t)-\bar{y}(t)\|\end{pmatrix}$$

By the condition (11) and the definition of stability, we obtain that

$$\begin{pmatrix}\|x(t)-\bar{x}(t)\| \\ \|y(t)-\bar{y}(t)\|\end{pmatrix} \le \begin{pmatrix}\epsilon_1 \\ \epsilon_2\end{pmatrix},\ \delta_1,\delta_2 > 0$$

So, that the solution of (1) is stable for all $t \in [0, T]$.

4. Existence and Uniqueness Solution of (1)

In this section, we prove the existence uniqueness theorem of the problem (1) by using Banach fixed point theorem.

Theorem 5 (Existence and Uniqueness Theorem): Let the vector functions $f(t, x, y, z)$ and $g(t, x, y, v)$ in the problem are defined and continuous on the domain and satisfy all conditions of theorem 2**,** then the problem has a unique continuous solution on the domain.

Proof: Let $(C[0,T], \|.\|)$ be Banach space and T be a mapping on $C[0, T]$ as follows:

$$T^*x(t) = x_0 + \int_0^t e^{A(t-s)}[f(s, x(s), y(s), \int_{-\infty}^{s} R(s-\tau)x(\tau)d\tau]ds \quad ...(41)$$

and

$$T^*y(t) = y_0 + \int_0^t e^{B(t-s)}[g(s, x(s), y(s), \int_{-\infty}^{s} H(s-\tau)y(\tau)d\tau]ds \quad ...(42)$$

Since $f(t, x, y, z)$ and $g(t, x, y, v)$ are continuous in the interval $[0, T]$ and x_0, y_0 are fixed points then

$$\int_0^t e^{A(t-s)}[f(s, x(s), y(s), \int_{-\infty}^{s} R(s-\tau)x(\tau)d\tau]ds$$

and

$$\int_0^t e^{B(t-s)}[g(s, x(s), y(s), \int_{-\infty}^{s} H(s-\tau)y(\tau)d\tau]ds$$

Are continuous functions on $C[0, T]$

Therefore, $T^*x(t), T^*y(t) \in C[0, T]$

Let $x(t), x^*(t), y(t), y^*(t) \in C[0, T]$ then

$$\|T^*x(t) - T^*x^*(t)\|$$
$$\leq \max_{t\in[0,T]}[\int_0^t \|e^{A(t-s)}\| [K_1\|x(s) - x^*(s)\| + K_2\|y(s) - y^*(s)\| + K_3\frac{cT^{\alpha-1}}{2\lambda}\|x(s) - x^*(s)\|]]ds$$

Therefore

$$\|T^*x(t) - T^*x^*(t)\| \leq E_1\|x(t) - x^*(t)\| + E_2\|y(t) - y^*(t)\| \quad ...(43)$$

And by the similar way, we get

$$\|T^*y(t) - T^*y^*(t)\|$$
$$\leq \max_{t\in[0,T]}[\int_0^t \|e^{B(t-s)}\| [L_1\|x(s) - x^*(s)\| + L_2\|y(s) - y^*(s)\| + L_3\frac{dT^{\beta-1}}{2\lambda}\|y(s) - y^*(s)\|]]ds$$

Hence

$$\|T^*y(t) - T^*y^*(t)\| \leq E_3\|x(t) - x^*(t)\| + E_4\|y(t) - y^*(t)\| \quad ...(44)$$

Rewrite the inequalities (43) and (44) in a vector form:

$$\begin{pmatrix}\|T^*x(t)-T^*x^*(t)\| \\ \|T^*y(t)-T^*y^*(t)\|\end{pmatrix} \le \begin{pmatrix}E_1 & E_2 \\ E_3 & E_4\end{pmatrix}\begin{pmatrix}\|x(t)-x^*(t)\| \\ \|y(t)-y^*(t)\|\end{pmatrix}$$

By the condition (11), then T^* is a contraction mapping on $C[0, T]$.

Therefore,

$$T^*x(t) = x(t) = x_0 + \int_0^t e^{A(t-s)}[f(s,x(s),y(s),\int_{-\infty}^{s} R(s-\tau)\,x(\tau)d\tau]ds$$

and

$$T^*y(t) = y(t) = y_0 + \int_0^t e^{B(t-s)}[g(s,x(s),y(s),\int_{-\infty}^{s} H(s-\tau)y(\tau)\,d\tau]ds$$

and hence $\begin{pmatrix}x(t) \\ y(t)\end{pmatrix}$ is a unique solution of (1).

Conclusion

This paper provided the existence, uniqueness and stability solutions of new integro-differential equations of the first order by using both method Picard approximation and Banach fixed point theorem. Theorems on the existence, uniqueness and stability solutions are established under some necessary and sufficient conditions on compact spaces.

Remark: The Picard approximation method given global solution but Banach fixed point theorem give us the local solution of integro-differential equations of the first order (1).

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